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Ground State and Resonances in the Standard Model of the Non-Relativistic QED

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Abstract We prove existence of a ground state and resonances in the standard model of the non-relativistic quantum electro-dynamics (QED). To this end we introduce a new canonical transformation of QED Hamiltonians and use the spectral renormalization group technique with a new choice of Banach spaces.

Keywords Non-relativistic QED · Quantum electro-dynamics · Spectrum · Ground state · Resonances · Renormalization group

1 Introduction

Problem and Outline of the Results Non-relativistic quantum electro-dynamics (QED) describes processes arising from interaction of the quantized electro-magnetic field with non-relativistic matter, such as emission and absorption of radiation by atoms and molecules. The mathematical framework of this theory is well established. It is given in terms of the time-dependent Schrödinger equation,

$$i\partial_t \psi = H_g^{SM} \psi,$$

where ψ is a differentiable path in the Hilbert space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f$, which is the tensor product of the state spaces of the matter system and the quantized electromagnetic field, and

To Jürg and Tom, with admiration.

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H_g^{SM} is the standard quantum Hamiltonian on $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f$, given by[1]

$$H_g^{SM} = \sum_{j=1}^{n} \frac{1}{2m_j}(i\nabla_{x_j} + gA(x_j))^2 + V(x) + H_f. \tag{1.1}$$

Here the superindex SM stands for 'standard model', m_j and x_j, $j = 1, \dots, n$, are the particle masses and positions, $x = (x_1, \dots, x_n)$, $V(x)$ is the total potential affecting particles,[2] $g > 0$ is a coupling constant related to the particle charge, $A(y)$ is the electromagnetic vector potential and H_f is the photon Hamiltonian. To have a self-adjoint and bounded from below quantum Hamiltonian we have to subject $A(y)$ to an ulraviolet (UV) cut-off.[3] The notions above and the remaining symbols are explained in detail below.

If we fix the particle potential $V(x)$ (e.g. taking it to be the total Coulomb potential), then the Hamiltonian (1.1) depends on two free parameters, the coupling constant g and the ultraviolet cut-off, not displayed here. As was mentioned above, g is related to the particle (electron) charges and the ultraviolet cut-off, to the particle (electron) renormalized mass (see [6, 18, 33, 34, 46, 53, 54], and [59] for a recent review).

For a large class of potentials $V(x)$, including Coulomb potentials, and under an ultraviolet cut-off, the operator H_g^{SM} is self-adjoint (see e.g. [13, 42]). The stability of the system under consideration is equivalent to the statement of existence of the ground state of H_g^{SM}, i.e. an eigenfunction with the smallest possible energy. The physical phenomenon of radiation is expressed mathematically as emergence of resonances out of excited states of a particle system due to coupling of this system to the quantum electro-magnetic field. We define the resonances and discuss their properties below.

In this paper we prove existence of the ground state and resonance states of H_g^{SM} originating from the ground state and from excited states of the particle system. Our approach provides also an effective way to compute the ground states and resonance states and their eigenvalues. We do not impose any extra conditions on H_g^{SM}, except for smallness of the coupling constant g and an ultraviolet cut-off in the interaction.

The standard model has been extensively studied in the last decade, see the book [59] and reviews [3, 28, 38, 43, 44] and references therein for a partial list of contributions and references and references [8, 9, 16, 17, 19, 20, 24, 35, 57] for some of more recent contributions.

The existence (and uniqueness) of the ground state was proven by compactness techniques in [4, 13, 31, 39–41, 45, 52] and in a constructive way, in [7].[4] The existence of the resonances was proven so far only for confined potentials (see [11, 12] and, for a book exposition, [32]).[5]

Our proof contains two new ingredients: a new canonical transformation of the Hamiltonian H_g^{SM} (which we call the generalized Pauli-Fierz transformation, Sect. 2) and new—momentum anisotropic—Banach spaces for the spectral renormalization group (RG) which allow us to control the RG flow for more singular coupling functions. (In the terminology of

[1] For discussion of physics emerging out of this Hamiltonian see [21, 22]. To simplify the exposition we omitted the interaction of the spin with magnetic field $\sum_{j=1}^{n} \frac{g}{2m_j}\sigma_j \cdot \mathrm{curl} A(x_j)$. It can be easily incorporated into our analysis.

[2] It could be helpful to think about the particles as electrons in an external field of static nuclei.

[3] For a given quantum model the UV cut-off is defined by an energy scale on which this model is applicable. In our case, the relevant energy scale is the characteristic energies of the particle motion.

[4] Analyticity of the ground state eigenvalues in parameters was proven in [30].

[5] Note that the papers [13, 31, 39, 45, 52] include the interaction of the spin with magnetic field in the Hamiltonian, while the present paper omits it.

the RG approach the perturbation in (1.1) is marginal (cf. critical nonlinearities in nonlinear PDEs), which is notoriously hard to treat; see a discussion below.) A part of this paper which deals with adapting and clarifying some points of the RG technique for the present situation (see Appendix B) is rather technical but can be used in other problems of non-relativistic QED.

Standard Model We now describe the standard model of non-relativistic QED. We use the units in which the Planck constant divided by 2π, the speed of light and the electron mass are equal to $1(\hbar = 1,\ c = 1$ and $m = 1)$. In these units the electron charge is equal to $-\sqrt{\alpha}$, where $\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$ (the fine-structure constant) and the distance, time and energy are measured in units of $\hbar/mc = 3.86 \times 10^{-11}$cm, $\hbar/mc^2 = 1.29 \times 10^{-21}$ s and $mc^2 = 0.511$ MeV, respectively (natural units).

We consider the matter system consisting of n charged particles interacting between themselves and with external fields and with a quantized electromagnetic field. The Hamiltonian operator of the particle system alone is given by

$$H_p := -\sum_{j=1}^{n} \frac{1}{2m_j} \Delta_{x_j} + V(x), \tag{1.2}$$

where Δ_{x_j} is the Laplacian in the variable x_j and, recall, $V(x)$ is the total potential of the particle system. This operator acts on a Hilbert space of the particle system, denoted by $\mathcal{H}_p$, which is either $L^2(\mathbb{R}^{3n})$ or a subspace of this space determined by a symmetry group of the particle system. We assume that $V(x)$ is real and s.t. the operator H_p is self-adjoint on the domain of $\sum_{j=1}^{n} \frac{1}{2m_j} \Delta_{x_j}$.

The quantized electromagnetic field is described by the *quantized vector potential*

$$A(y) = \int (e^{iky} a(k) + e^{-iky} a^*(k)) \chi(k) \frac{d^3k}{\sqrt{|k|}}, \tag{1.3}$$

written in the Coulomb gauge ($\mathrm{div} A(y) = 0$). Here χ is an *ultraviolet cut-off*: $\chi(k) = \frac{1}{(2\pi)^3\sqrt{2}}$ in a neighborhood of $k = 0$ and it vanishes sufficiently fast at infinity (we comment of this below). The dynamics of the quantized electromagnetic field is given through the quantum Hamiltonian

$$H_f = \int d^3k \omega(k) a^*(k) \cdot a(k), \tag{1.4}$$

where $\omega(k) = |k|$ is the dispersion law connecting the energy of the field quantum with its wave vector k. Both, $A(y)$ and H_f, act on the Fock space $\mathcal{H}_f \equiv \mathcal{F}$. Thus the Hilbert space of the total system is $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}$.

Above, $a^*(k)$ and $a(k)$ denote the creation and annihilation operators on $\mathcal{F}$. The families $a^*(k)$ and $a(k)$ are operator-valued generalized, transverse vector fields:

$$a^\#(k) := \sum_{\lambda \in \{-1,1\}} e_\lambda(k) a^\#_\lambda(k),$$

where $e_\lambda(k)$ are polarization vectors, i.e. orthonormal vectors in $\mathbb{R}^3$ satisfying $k \cdot e_\lambda(k) = 0$, and $a^\#_\lambda(k)$ are scalar creation and annihilation operators satisfying canonical commutation relations. The right side of (1.4) can be understood as a weak integral. See Appendix C for a brief review of definitions of the Fock space, the creation and annihilation operators and the operator H_f.

The Hamiltonian of the total system, matter and radiation field, is given by (1.1). First, we consider (1.1) for an atom or molecule. Then, in the natural units, $g = \sqrt{\alpha}$ and $V(x)$, the total Coulomb potential of the particle system, is proportional to α. Rescaling $x \to \alpha^{-1}x$ and $k \to \alpha^2 k$ we arrive at (1.1) with $g := \alpha^{3/2}$, $V(x)$ of the order $O(1)$[6] and $A(x)$ replaced by $A'(x)$, where $A'(x) = A(\alpha x)|_{\chi(k)\to\chi'(k)}$, and where $\chi'(k) := \chi(\alpha^2 k)$ (see [10, 13]). After that we drop the prime in the vector potential $A'(x)$ and the ultraviolet cut-off $\chi'(x)$ (see a discussion of the latter below). Finally, we relax the restriction on $V(x)$ by considering the standard generalized n-body potentials (see e.g. [49]):

(V) $V(x) = \sum_i W_i(\pi_i x)$, where π_i are a linear maps from $\mathbb{R}^{3n}$ to $\mathbb{R}^{m_i}$, $m_i \le 3n$ and W_i are Kato-Rellich potentials (i.e. $W_i(\pi_i x) \in L^{p_i}(\mathbb{R}^{m_i}) + (L^\infty(\mathbb{R}^{3n}))_\varepsilon$ with $p_i = 2$ for $m_i \le 3$, $p_i > 2$ for $m_i = 4$ and $p_i \ge m_i/2$ for $m_i > 4$, see [47, 58]).

Under the assumption (V), the operator H_g^{SM} is self-adjoint. In order to tackle the resonances we choose the ultraviolet cut-off, $\chi(k)$, so that

The function $\theta \to \chi(e^{-\theta}k)$ has an analytic continuation from the real axis, $\mathbb{R}$, to the strip $\{\theta \in \mathbb{C} | |\operatorname{Im}\theta| < \pi/4\}$ as a $L^2 \bigcap L^\infty(\mathbb{R}^3)$ function,

e.g. $\chi(k) = e^{-|k|^2/\kappa^2}$. Furthermore, we assume that the potential, $V(x)$, satisfies the condition:

(DA) The the particle potential $V(x)$ is dilation analytic in the sense that the operator-function $\theta \to V(e^\theta x)(-\Delta + 1)^{-1}$ has an analytic continuation from the real axis, $\mathbb{R}$, to the strip $\{\theta \in \mathbb{C} | |\operatorname{Im}\theta| < \theta_0\}$ for some $\theta_0 > 0$.

In order not to deal with the problem of center-of-mass motion which is not essential in the present context, we assume that either some of the particles (nuclei) are infinitely heavy or the system is placed in a binding, external potential field. This means that the operator H_p has isolated eigenvalues below its essential spectrum. However, we expect that the techniques developed in this paper can be extended to translationally invariant particle systems (see [2, 23, 55]).

Ultra-Violet Cut-Off Finally, we comment on the ultra-violet cut-off $\chi(k)$ introduced in (1.3). This cut-off makes the model well-defined. Assuming χ decays on the scale κ, in order to correctly describe the phenomena of interest, such as emission and absorption of electromagnetic radiation, i.e. for optical and rf modes, we have to assume that the cut-off energy, $\hbar c\kappa$, is much greater than the characteristic energies of the particle motion. (We reintroduced the Planck constant, $\hbar$, and speed of light, c, for a moment.) The latter motion takes place on the energy scale of the order of the ionization energy, i.e. of the order $\alpha^2 mc^2$. Thus we have to assume $\alpha^2 mc^2 \ll \hbar c\kappa$.

On the other hand, for energies higher than the rest energy of the electron (mc^2) the relativistic effects, such as electron-positron pair creation, vacuum polarization and relativistic recoil, take place. Thus it makes sense to assume that $\hbar c\kappa \ll mc^2$. Combining the last two conditions we arrive at the restriction $\alpha^2 mc^2 \ll \hbar c\kappa \ll mc^2$ or $\alpha^2 mc/\hbar \ll \kappa \ll mc/\hbar$. In our units this reads

$$\alpha^2 \ll \kappa \ll 1.$$

[6]In the case of a molecule in the Born-Oppenheimer approximation, the resulting $V(x)$ also depends on the rescaled coordinates of the nuclei.

After the rescaling $x \to \alpha^{-1}x$ and $k \to \alpha^2 k$ performed above the new cut-off momentum scale, $\kappa' = \alpha^{-2}\kappa$, satisfies

$$1 \ll \kappa' \ll \alpha^{-2},$$

which is easily accommodated by our estimates (e.g. we can have $\kappa = O(\alpha^{-1/3})$). Thus we can assume for simplicity that χ is fixed.

Resonances We define the resonances for the Hamiltonian H_g^{SM} as follows. Consider the dilations of particle positions and of photon momenta:

$$x_j \to e^{\theta} x_j \quad \text{and} \quad k \to e^{-\theta} k,$$

where θ is a real parameter. Such dilations are represented by the one-parameter group of unitary operators, U_θ, on the total Hilbert space $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}$ of the system (see Sect. 3). Now, for $\theta \in \mathbb{R}$ we define the deformation family

$$H_{g\theta}^{SM} := X_\theta H_g^{SM} X_\theta^{-1}, \tag{1.5}$$

where $X_\theta := U_\theta e^{-igF}$ with F, the self-adjoint operator defined in Sect. 2. The transformation $H_g^{SM} \to e^{-igF} H_g^{SM} e^{igF}$ is a generalization of the well-known Pauli-Fierz transformation. Note that the operator-family X_θ has the following two properties needed in order to establish the desired properties of the resonances:

(a) X_θ are unitary for $\theta \in \mathbb{R}$;
(b) $X_{\theta_1+\theta_2} = U_{\theta_1} X_{\theta_2}$ where U_θ are unitary for $\theta \in \mathbb{R}$.

It is easy to show (see Sect. 3) that, due to Condition (DA), the family $H_{g\theta}^{SM}$ has an analytic continuation in θ to the disc $D(0, \theta_0)$, as a type A family in the sense of Kato ([50]). A standard argument shows that the real eigenvalues of $H_{g\theta}^{SM}$, $\operatorname{Im}\theta > 0$, coincide with eigenvalues of H_g^{SM} and that complex eigenvalues of $H_{g\theta}^{SM}$, $\operatorname{Im}\theta > 0$, lie in the complex half-plane $\mathbb{C}^-$. We show below that the complex eigenvalues of $H_{g\theta}^{SM}$, $\operatorname{Im}\theta > 0$, are locally independent of θ. We call such eigenvalues the *resonances* of H_g^{SM}.

It is clear from the definition that the notion of resonance extends that of eigenvalue and under small perturbations embedded eigenvalues turn generally into resonances. Correspondingly, the resonances share two 'physical' manifestations of eigenvalues, as poles of the resolvent and frequencies of time-periodic and spatially localized solutions of the time-dependent Schrödinger equation, but with a caveat. To explain the first property, we use the Combes argument which goes as follows. By the unitarity of $X_\theta := U_\theta e^{-igF}$ for real θ,

$$\langle \Psi, (H_g^{SM} - z)^{-1} \Phi \rangle = \langle \Psi_{\bar{\theta}}, (H_{g\theta}^{SM} - z)^{-1} \Phi_\theta \rangle, \tag{1.6}$$

where $\Psi_\theta = X_\theta \Psi$, etc., for $\theta \in \mathbb{R}$ and $z \in \mathbb{C}^+$. Assume now that Ψ_θ and Φ_θ have analytic continuations into a complex neighborhood of $\theta = 0$. Then the r.h.s. of (1.6) has an analytic continuation in θ into a complex neighborhood of $\theta = 0$. Since (1.6) holds for real θ, it also holds in the above neighborhood. Fix θ on the r.h.s. of (1.6), with $\operatorname{Im}\theta > 0$. The r.h.s. of (1.6) can be analytically extended across the real axis into the part of the resolvent set of $H_{g\theta}^{SM}$ which lies in $\overline{\mathbb{C}^-}$ and which is connected to $\mathbb{C}^+$. This yields an analytic continuation of the l.h.s. of (1.6). The real eigenvalues of $H_{g\theta}^{SM}$ give real poles of the r.h.s. of (1.6) and therefore they are the eigenvalues of H_g^{SM}. The complex eigenvalues of $H_{g\theta}^{SM}$, which are at the resonances of H_g^{SM}, yield complex poles of the r.h.s. of (1.6) and therefore they are poles of the meromorphic continuation of the l.h.s. of (1.6) across the spectrum of H_g^{SM} onto the second Riemann sheet. This pole structure is observed physically as bumps in the scattering cross-section or poles in the scattering matrix. There are some subtleties involved which we explain below.

The second manifestation of resonances alluded to above is as metastable states (metastable attractors of system's dynamics). Namely, one expects that the ground state is asymptotically stable and the resonance states are (asymptotically) metastable, i.e. attractive for very long time intervals. More specifically, let z_*, $\operatorname{Im} z_* \le 0$, be a ground state or resonance eigenvalue. One expects that for an initial condition, ψ_0, localized in a small energy interval around the ground state or resonance energy, $\operatorname{Re} z_*$, the solution, ψ, of the time-dependent Schrödinger equation, $i\partial_t \psi = H_g^{SM}\psi$, is of the form

$$e^{-iH_g^{SM}t}\psi_0 = e^{-iz_*t}\phi_* + O_{\mathrm{loc}}(t^{-\alpha}) + O_{\mathrm{res}}(g^\beta), \tag{1.7}$$

for some $\alpha, \beta > 0$ (depending on ψ_0). Here ϕ_* is either the ground state (if z_* is the ground state energy) or an excited state of the unperturbed system (if z_* is a resonance eigenvalue); the error term $O_{\mathrm{loc}}(t^{-\alpha})$ satisfies $\|(\mathbf{1} + |T|)^{-\nu} O_{\mathrm{loc}}(t^{-\alpha})\| \le Ct^{-\alpha}$, where T is the generator of the group U_θ, with an appropriate $\nu > 0$; and the error term $O_{\mathrm{res}}(g^\beta)$ is absent in the ground state case. The reason for the latter is that, unlike bound states, there is no 'canonical' notion of the resonance state.

The asymptotic stability of the ground state is equivalent to the statement of local decay. Its proof was completed recently in [25, 27] (see [13, 14] for complementary results). A statement involving survival probabilities of excited states which is related to the metastability of the resonances is proven in [1] using the results of this paper (see [36] for related results and [13, 51, 56] for partial results).

The dynamical picture of the resonance described above implies that the imaginary part of the resonance eigenvalue, called the resonance width, can be interpreted as the decay rate probability, and its reciprocal, as the life-time, of the resonance.

Main Results Let $\epsilon_0^{(p)} < \epsilon_1^{(p)} < \cdots$ be the isolated eigenvalues of the particle Hamiltonian H_p. In what follows we fix an energy $\nu \in (\epsilon_0^{(p)}, \inf \sigma_{ess}(H_p))$ below the ionization threshold $\inf \sigma_{ess}(H_p)$ and denote $\epsilon_{gap}^{(p)} \equiv \epsilon_{gap}^{(p)}(\nu) := \min\{|\epsilon_i^{(p)} - \epsilon_j^{(p)}| \mid i \ne j, \epsilon_i^{(p)}, \epsilon_j^{(p)} \le \nu\}$ and $j(\nu) := \max\{j : \epsilon_j^{(p)} \le \nu\}$.

We now state the main results of this paper.

Theorem 1.1 *Assume Condition* (*V*). *Fix* $e_0^{(p)} < \nu < \inf \sigma_{ess}(H_p)$ *and let* $g \ll \min(\epsilon_{gap}^{(p)}(\nu), \sqrt{\epsilon_{gap}^{(p)}(\nu)\tan(\theta_0/2)})$. *Then*

(i) *Each eigenvalue*, $\epsilon_j^{(p)}$, *of* $H_{g=0}^{SM}$, *which is less than* ν, *turns into resonance and/or bound state eigenvalues*, $\epsilon_{j,k}$, *of* H_g^{SM}, $g \ne 0$;

(ii) $\forall j$, $\epsilon_{j,k} = \epsilon_j^{(p)} + O(g^2)$ *and the total multiplicity of* $\epsilon_{j,k} \forall k$ *equals the multiplicity of* $\epsilon_j^{(p)}$;

(iii) H_g^{SM} *has a ground state, originating from a ground state of* $H_{g=0}^{SM}$;

(iv) $\epsilon_{j,k}$*'s are independent of* θ, $0 < \operatorname{Im}\theta \ge \theta_0$.

The statements concerning the excited states are proven under additional Condition (*DA*).

By statement (ii), we have $\epsilon_0 := \inf \sigma(H_g^{SM})$. Let

$$S_{j,k} := \left\{ z \in \mathbb{C} \,\Big|\, \frac{1}{2}\operatorname{Re}(e^{\theta}(z - \epsilon_{j,k})) \geq |\operatorname{Im}(e^{\theta}(z - \epsilon_{j,k}))| \right\}. \tag{1.8}$$

Information about meromorphic continuation of the matrix elements of the resolvent and its behavior near the resonances is given in the next theorem.

Theorem 1.2 *Assume $g \ll \epsilon_{gap}^{(p)}(\nu)$ and Conditions (V) and (DA). Then for a dense set (defined in (1.9) below) of vectors Ψ and Φ, the matrix elements $F(z, \Psi, \Phi) := \langle \Psi, (H_g^{SM} - z)^{-1}\Phi\rangle$ of the resolvent of H_g^{SM} have meromorphic continuations from $\mathbb{C}^+$ across the interval (ϵ_0, ν) of the essential spectrum of H_g^{SM} into the domain $\{z \in \mathbb{C}^- \mid \epsilon_0 < \operatorname{Re} z < \nu\}$, with the wedges $S_{j,k}, 0 \leq j \leq j(\nu)$, deleted. Furthermore, this continuation has poles at $\epsilon_{j,k}$ in the sense that $\lim_{z\to\epsilon_{j,k}}(\epsilon_{j,k} - z)F(z, \Psi, \Phi)$ is finite and, for a finite-dimensional subspace of Ψ's and Φ's, nonzero.*

Discussion

(i) Condition (DA) could be weakened considerably so that it is satisfied by the potential of a molecule with fixed nuclei (cf. [49]).

(ii) Generically, excited states turn into the resonances, not bound states. A condition which guarantees that this happens is the Fermi Golden Rule (FGR) (see [13]). It expresses the fact that the coupling of unperturbed embedded eigenvalues of H_0^{SM} to the continuous spectrum is effective in the second order of the perturbation theory. It is generically satisfied.

(iii) With a little more work one can establish an explicit restriction on the coupling constant g in terms of the particle energy difference $e_{gap}^{(p)}(\nu)$ and appropriate norms of the coupling functions.

(iv) The second theorem implies the absolute continuity of the spectrum and its proof gives also the limiting absorption principle in the interval (ϵ_0, ν), but these results have already been proven by the spectral deformation and commutator techniques [13, 14, 25].

(v) The meromorphic continuation in question is constructed in terms of matrix elements of the resolvent of a complex deformation, $H_{g,\theta}^{SM}$, $\operatorname{Im}\theta > 0$, of the Hamiltonian H_g^{SM}.

(vi) The proof of Theorem 1.1 gives fast convergent expressions in the coupling constant g for the ground state energy and resonances.

The main new result of this work is the existence of resonances and an algorithm for their computation.

The dense set mentioned in the Theorem 1.2 is defined as

$$\mathcal{D} := \bigcup_{n>0,a>0} \operatorname{Ran}\big(\chi_{N\leq n}\chi_{|T|\leq a}\big). \tag{1.9}$$

Here $N = \int d^3k a^*(k)a(k)$ is the photon number operator and, recall, T denotes the self-adjoint generator of the one-parameter group $U_\theta, \theta \in \mathbb{R}$ (see Sect. 3). Since N and T commute, this set is dense. We claim that for any $\psi \in \mathcal{D}$, the family $U_\theta e^{-igF(x)}\psi$ has an analytic continuation from $\mathbb{R}$ to the complex disc $D(0, \theta_0)$. Indeed, by the construction in the next section, the family $F_\theta(x) := U_\theta F(x)U_\theta^{-1}$ has an analytic continuation from $\mathbb{R}$ to the complex disc $D(0, \theta_0)$. For θ complex this continuation is a family of non-self-adjoint operators.

However, the exponential $e^{-igF_\theta(x)}$ is well defined on the dense domain $\bigcup_{n<\infty} \mathrm{Ran}\chi_{N\le n}$. Since

$$U_\theta e^{igF(x)}\psi = e^{igF_\theta(x)}\chi_{N\le n}U_\theta\chi_{|T|\le a}\psi$$

for some n and a, s.t. $\chi_{N\le n}\chi_{|T|\le a}\psi=\psi$, the family $U_\theta e^{-igF(x)}\psi$ has an analytic continuation in θ from $\mathbb{R}$ to $D(0,\theta_0)$.

Infrared Problem As is shown in Theorem 1.1 and is understood in Physics on the basis of formal—but rather non-trivial—perturbation theory, the resonances arise from the eigenvalues of the non-interacting Hamiltonian H_0^{SM}. To find the spectrum of H_0^{SM} one verifies that H_f defines a positive, self-adjoint operator on $\mathcal{F}$ with purely absolutely continuous spectrum, except for a simple eigenvalue 0 corresponding to the vacuum eigenvector Ω (see Appendix C). Thus, for $g=0$ the low energy spectrum of the operator H_0^{SM} consists of branches $[\epsilon_i^{(p)},\infty)$ of absolutely continuous spectrum and of the eigenvalues $\epsilon_i^{(p)}$'s, sitting at the continuous spectrum 'thresholds' $\epsilon_i^{(p)}$'s. The absence of gaps between the eigenvalues and thresholds is a consequence of the fact that the photons are massless. This leads to hard problems in perturbation theory, known collectively as the *infrared problem*.

This situation is quite different from the one in Quantum Mechanics (e.g. Stark effect or tunneling decay) where the resonances are isolated eigenvalues of complexly deformed Hamiltonians. This makes the proof of their existence and establishing their properties, e.g. independence of θ (and, in fact, of the transformation group X_θ), relatively easy. In the non-relativistic QED (and other massless theories), giving meaning of the resonance poles and proving independence of their location of θ is a rather involved matter. We illustrate it on the proof of the statement (1.7). To this end we use the formula

$$e^{-iHt}f(H)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\lambda f(\lambda)e^{-i\lambda t}\,\mathrm{Im}(H-\lambda-i0)^{-1}$$

(see e.g. [58]) connecting the propagator and the resolvent. For the ground state, the absolute continuity of the spectrum outside the ground state energy, or a stronger property of the limiting absorption principle, suffices to establish the result in question. In the resonance case, one observes the fact that the meromorphic continuation of matrix elements of the resolvent (on an appropriate dense set of vectors) to the second Riemann sheet has poles at resonances and one performs a suitable deformation of the contour of integration in the formula above to pick up the contribution of these poles (see e.g. [48]). This works when the resonances are isolated (see [48, 49]). In the present case, proving (1.7) is a subtle problem.

Resonance Poles Can we make sense of the resonance poles in the present context? The answer to this question is obtained in [1], where it is shown, assuming the results of this paper, that for each Ψ and Φ from a dense set of vectors, the meromorphic continuation, $F(z,\Psi,\Phi)$, of the matrix element $\langle\Psi,(H_g^{SM}-z)^{-1}\Phi\rangle$, described above, is of the following form:

$$F(z,\Psi,\Phi)=(\epsilon_{j,k}-z)^{-1}p(\Psi,\Phi)+r(z,\Psi,\Phi), \tag{1.10}$$

near the resonance ϵ_j of H_g^{SM}. Here p and $r(z)$ are sesquilinear forms in Ψ and Φ with $r(z)$, analytic in $z\in Q:=\{z\in\mathbb{C}^-\mid \epsilon_0<\mathrm{Re}\,z<\nu\}/\bigcup_{j\le j(\nu),k}S_{j,k}$ and bounded on the intersection of a neighborhood of $\epsilon_{j,k}$ with Q as

$$|r(z,\Psi,\Phi)|\le C_{\Psi,\Phi}|\epsilon_{j,k}-z|^{-\gamma}\text{ for some }\gamma<1.$$

Moreover, $p \neq 0$ at least for one pair of vectors Ψ and Φ and $p = 0$ for a dense set of vectors Ψ and Φ in a finite co-dimension subspace. The multiplicity of a resonance is the rank of the residue at the pole. *The next important problem is to connect the ground state and resonance eigenvalues to poles of the scattering matrix.*

Approach To prove Theorems 1.1 and 1.2 we apply the spectral renormalization group (RG) method ([5, 11, 12, 26, 30]) to the Hamiltonians $H^{SM}_{g\theta=0} = e^{-igF} H^{SM}_g e^{igF}$ (the ground state case) and $H^{SM}_{g\theta}$, $\operatorname{Im}\theta > 0$, (the resonance case). Note that the version of RG needed in this work uses new—anisotropic—Banach spaces of operators, on which the renormalization group acts. It is developed in [26]. Using the RG technique we describe the spectrum of the operator $H^{SM}_{g\theta}$ in $\{z \in \mathbb{C}^- | \epsilon_0 < \operatorname{Re} z < \nu\}$ from which we derive Theorems 1.1 and 1.2.

In the terminology of the Renormalization Group approach the perturbation in (1.1) is marginal (similar to critical nonlinearities in nonlinear PDEs). This leads to the presence of the second zero eigenvalue in the spectrum of the linearized RG flow (note that there is no spectral gap in the linearized RG flow). This case is notoriously hard to treat as one has to understand the dynamics on the implicitly defined central manifold. The previous works [11, 12] remove it by either assuming the non-physical infrared behavior of the vector potential by replacing $|k|^{-1/2}$ in the vector potential (1.3) by $|k|^{-1/2+\varepsilon}$, with $\varepsilon > 0$, or by assuming presence of a strong confining external potential so that $V(x) \geq c|x|^2$ for x large. Our work shows that in non-relativistic QED one can overcome this problem by suitable canonical transformation and choice of the Banach space.

Our approach is also applicable to Nelson's model describing interaction of particles with massless lattice excitations (phonons) described by a quantized, massless, Boson field (see Appendix D), and Theorems 1.1 and 1.2 are still valid if replace there the operator H^{SM}_g by the Hamiltonian H^N_g for this model. (In this case we recover earlier results.) In fact, we consider a class of generalized particle-field operators (introduced in Sect. 4) which contains both, operators H^{PF}_g and H^N_g.

Organization of the Paper In Sect. 2 we introduce the generalized Pauli-Fierz transformation ($H^{SM}_g \to e^{-igF} H^{SM}_g e^{igF} =: H^{PF}_g$) and in Sect. 3, the complex deformation of quantum Hamiltonians. In Sect. 4 we introduce a class of generalized particle-field Hamiltonians and show that the Hamiltonian H^{PF}_g obtained in Sect. 2 and the Hamiltonian H^N_g as well as their dilation deformations belong to this class. In the rest of the paper we study the Hamiltonians from the class introduced and derive Theorems 1.1 and 1.2 from the results about these Hamiltonians. In Sect. 5 we introduce an isospectral Feshbach-Schur map and use it to map the generalized particle-field Hamiltonians into Hamiltonians acting only on the field Hilbert space—Fock space (elimination of particle and high photon energy degrees of freedom). The image of this map is shown in Sect. 7 to belong to a certain neighborhood in the Banach spaces introduced in Sect. 6. The latter spaces are an anisotropic—in the momentum representation—modification of the Banach spaces used in [5, 11, 12]. In Sect. 8 we use the results of [26] on the spectral renormalization group (cf. [5, 11, 12]) to describe the spectrum of generalized particle-field Hamiltonians. Finally, in Sect. 9 we prove Theorems 1.1 and 1.2. In Appendix A we recall some properties of the Feshbach-Schur map and in Appendix B we prove the main result of Sect. 6. The results of both appendices are close to certain results from [5, 26, 30], but there are a few important differences. The main ones are that we have to deal with unbounded interactions and, more importantly, with momentum-anisotropic spaces. Some basic facts about Fock spaces and creation and annihilation operators on them are collected in Appendix C and in Appendix D we describe the Nelson Hamiltonians and their dilation deformations. In order to simplify the notation and exposition below we assume that the *particle eigenvalues*, $\epsilon_j^{(p)}$, we deal with, are *non-degenerate* and consequently the subindex k can be omitted in $\epsilon_{j,k}$ and $S_{j,k}$ which from now on are written as ϵ_j and S_j. In order to treat with degenerate eigenvalues we would have to deal with matrix-valued operators on the Fock space.

2 Generalized Pauli-Fierz Transformation

In order to simplify notation from now on we assume that the number of particles is $1, n = 1$. We also set the particle mass to 1, $m = 1$. The generalizations to an arbitrary number of particles is straightforward. We define the generalized Pauli-Fierz transformation mentioned in the introduction:

$$H_g^{PF} := e^{-igF(x)} H_g^{SM} e^{igF(x)}, \tag{2.1}$$

where $F(x)$ introduced below. We call the resulting Hamiltonian the generalized Pauli-Fierz Hamiltonian. Now, $F(x)$ is the self-adjoint operator on the state space $\mathcal{H}$ given by

$$F(x) = \sum_\lambda \int (\bar{f}_{x,\lambda}(k) a_\lambda(k) + f_{x,\lambda}(k) a_\lambda^*(k)) \frac{d^3k}{\sqrt{|k|}}, \tag{2.2}$$

with the coupling function $f_{x,\lambda}(k)$ chosen as

$$f_{x,\lambda}(k) := e^{-ikx} \frac{\chi(k)}{\sqrt{|k|}} \varphi(|k|^{\frac{1}{2}} e_\lambda(k) \cdot x). \tag{2.3}$$

The function φ is assumed to be C^2, bounded, with bounded second derivative, and satisfying $\varphi'(0) = 1$. We assume also that φ has a bounded analytic continuation into the wedge $\{z \in \mathbb{C} | |\arg(z)| < \theta_0\}$. We compute

$$H_g^{PF} = \frac{1}{2}(p - gA_1(x))^2 + V_g(x) + H_f + gG(x) \tag{2.4}$$

where $A_1(x) = A(x) - \nabla F(x)$, $V_g(x) := V(x) + 2g^2 \sum_\lambda \int |k||f_{x,\lambda}(k)|^2 d^3k$ and

$$G(x) := -i \sum_\lambda \int |k|(\bar{f}_{x,\lambda}(k) a_\lambda(k) - f_{x,\lambda}(k) a_\lambda^*(k)) \frac{d^3k}{\sqrt{|k|}}. \tag{2.5}$$

(The terms gG and $V_g - V$ come from the commutator expansion $e^{-igF(x)} H_f e^{igF(x)} = -ig[F, H_f] - g^2[F, [F, H_f]]$.) Observe that the operator-family $A_1(x)$ is of the form

$$A_1(x) = \sum_\lambda \int (e^{ikx} a_\lambda(k) + e^{-ikx} a_\lambda^*(k)) \chi_{\lambda,x}(k) \frac{d^3k}{\sqrt{|k|}}, \tag{2.6}$$

where the coupling function $\chi_{\lambda,x}(k)$ is defined as follows

$$\chi_{\lambda,x}(k) := e_\lambda(k) e^{-ikx} \chi(k) - \nabla_x f_{x,\lambda}(k).$$

It satisfies the estimates

$$|\chi_{\lambda,x}(k)| \le \operatorname{const} \min(1, \sqrt{|k|}\langle x \rangle), \tag{2.7}$$

with $\langle x\rangle := (1+|x|^2)^{1/2}$, and

$$\int \frac{d^3k}{|k|}|\chi_{\lambda,x}(k)|^2 < \infty. \tag{2.8}$$

The fact that the operators A_1 and G have better infra-red behavior than the original vector potential A, is used in proving, with a help of a renormalization group, the existence of the ground state and resonances for the Hamiltonian H_g^{SM}.

We note that for the standard Pauli-Fierz transformation, the function $f_{x,\lambda}(k)$ is chosen to be $\chi(k)e_\lambda(k)\cdot x$, which results in the operator G (which in this case is proportional to (the electric field at $x=0$) $\cdot\, x$) growing as x.

We mention for further references that the operator (2.4) can be written as

$$H_g^{PF} = H_0^{PF} + I_g^{PF}, \tag{2.9}$$

where $H_0^{PF} = H_0 + 2g^2\sum_\lambda \int |k||f_{x,\lambda}(k)|^2 d^3k + g^2\sum_\lambda \int \frac{|\chi_\lambda(k)|^2}{|k|}d^3k$, with $H_0 := H_p + H_f$ and I_g^{PF} is defined by this relation. Note that the operator I_g^{PF} contains linear and quadratic terms in the creation and annihilation operators and that the operator H_0^{PF} is of the form $H_0^{PF} = H_p^{PF} + H_f$ where

$$H_p^{PF} := H_p + 2g^2\sum_\lambda \int |k||f_{x,\lambda}(k)|^2 d^3k + g^2\sum_\lambda \int \frac{|\chi_\lambda(k)|^2}{|k|}d^3k \tag{2.10}$$

with H_p given in (1.2).

Since the operator $F(x)$ in (2.1) is self-adjoint, the operators H_g^{SM} and H_g^{PF} have the same eigenvalues with closely related eigenfunctions and the same essential spectra.

3 Complex Deformation and Resonances

In this section we define complex transformation of the Hamiltonian under consideration which underpins the proof of the resonance part of Theorem 1.1 and the proof of Theorem 1.2. Let u_θ be the dilatation transformation on the one-photon space, i.e., $u_\theta\colon f(k)\to e^{-3\theta/2}f(e^{-\theta}k)$. Define the dilatation transformation, $U_{f\theta}$, on the Fock space, $\mathcal{H}_f\equiv\mathcal{F}$, by second quantizing u_θ: $U_{f\theta} := e^{iT\theta}$ where $T := \int a^*(k)ta(k)dk$ and t is the generator of the group u_θ (see Appendices C and D for the careful definition of the above integral). This gives, in particular,

$$U_{f\theta}a^*(f)U_{f\theta}^{-1} = a^*(u_\theta f). \tag{3.1}$$

Denote by $U_{p\theta}$ the standard dilation group on the particle space: $U_{p\theta}:\psi(x)\to e^{\frac{3}{2}\theta}\psi(e^\theta x)$ (remember that we assumed that the number of particles is 1). We define the dilation transformation on the total space $\mathcal{H}=\mathcal{H}_p\otimes\mathcal{H}_f$ by

$$U_\theta = U_{p\theta}\otimes U_{f\theta}. \tag{3.2}$$

For $\theta\in\mathbb{R}$ the above operators are unitary and map the domains of the operators below into themselves. Consequently, we can define the family of Hamiltonians originating from the Hamiltonian H_g^{SM} as

$$H_{g\theta}^{SM} := U_\theta e^{-igF(x)}H_g^{SM}e^{igF(x)}U_\theta^{-1}. \tag{3.3}$$

Under Condition (DA), there is a Type-A ([47, 50, 58]) family $H^{SM}_{g\theta}$ of operators analytic in the domain $|\operatorname{Im}\theta| < \theta_0$, which is equal to (3.3) for $\theta \in \mathbb{R}$ and s.t. $H^{SM*}_{g\theta} = H^{SM}_{g\bar{\theta}}$ and

$$H^{SM}_{g\theta} = U_{\operatorname{Re}\theta} H^{SM}_{gi\operatorname{Im}\theta} U^{-1}_{\operatorname{Re}\theta}. \tag{3.4}$$

Indeed, using the decomposition $H^{PF}_g = H^{PF}_p + H_f + I^{PF}_g$ (see (2.9)–(2.10)), we write for $\theta \in \mathbb{R}$

$$H^{SM}_{g\theta} = H^{SM}_{p\theta} \otimes \mathbf{1}_f + e^{-\theta}\mathbf{1}_p \otimes H_f + I^{SM}_{g\theta}, \tag{3.5}$$

where $H^{SM}_{p\theta} := U_{p\theta} H^{PF}_p U^{-1}_{p\theta}$ and $I^{SM}_{g\theta} := U_\theta I^{PF}_g U^{-1}_\theta$. It is not hard to compute that $H^{SM}_{p\theta} = -e^{-2\theta}\frac{1}{2}\Delta + V_g(e^\theta x)$, where

$$V_g(x) := V(x) + 2g^2 \sum_\lambda \int |k||f_{x,\lambda}(k)|^2 d^3k + g^2 \sum_\lambda \int \frac{|\chi_\lambda(k)|^2}{|k|} d^3k \tag{3.6}$$

with V given in (1.2). Furthermore, using (3.1) and the definitions of the interaction I^{PF}_g, we see that $I^{SM}_{g\theta}$ is obtained from I^{PF}_g by the replacement $a^\#(k) \to e^{-\frac{3\theta}{2}} a^\#(k)$ and, in the coupling functions only,

$$k \to e^{-\theta}k \text{ and } x \to e^\theta x. \tag{3.7}$$

By Condition (DA), the family (3.5) is well defined for all θ satisfying $|\operatorname{Im}\theta| < \theta_0$ and has all the properties mentioned after (3.3). Hence, for these θ, it gives the required analytic continuation of (3.3). We call $H^{SM}_{g\theta}$ with $\operatorname{Im}\theta > 0$ the complex deformation of H^{SM}_g.

Recall that we define the resonances of H^{SM}_g as the complex eigenvalues of $H^{SM}_{g\theta}$ with $\operatorname{Im}\theta > 0$. Thus to find resonances (and eigenvalues) of H^{SM}_g we have to locate complex (and real) eigenvalues of $H^{SM}_{g\theta}$ for some θ with $\operatorname{Im}\theta > 0$.

In Sects. 5–8 we prove the following result

Theorem 3.1 *Assume Conditions (V) and (DA) holds. Fix $e^{(p)}_0 < \nu < \inf\sigma_{ess}(H_p)$ and let $g \ll \epsilon^{(p)}_{gap}(\nu)$. Then the operators $H^{SM}_{g\theta}$, with $\operatorname{Im}\theta > 0$, have eigenvalues[7] ϵ_j, $j \le j(\nu)$, s.t. $\epsilon_j = \epsilon^{(p)}_j + O(g^2)$ and ϵ_j are independent of θ. The essential spectrum of $H^{SM}_{g\theta}$, $\operatorname{Im}\theta > 0$, is a subset of the set $\bigcup_{j\le j(\nu)} S_j$, where the sets S_j are given in (1.8).*

Theorem 3.1, together with the discussion in paragraphs containing (1.9)–(1.6) implies Theorems 1.1 and 1.2 (for the ground state part of Theorem 1.1 it contains unnecessary Condition (DA)).

Furthermore, one can show that the eigenvalues ϵ_j, $j \le j(\nu)$, have the following properties

(i) If the FGR condition is satisfied, then $\operatorname{Im}\epsilon_j = -g^2\gamma_j + O(g^4)$, where γ_j are given by the Fermi Golden Rule formula;
(ii) ϵ_j can be computed explicitly in terms of fast convergent expressions in the coupling constant g.

[7]Remember that we assume that the particle eigenvalues are non-degenerate and consequently the second subindex k in $\epsilon_{j,k}$ and $S_{j,k}$ drops out.

4 Generalized Particle-Field Hamiltonians

It is convenient to consider a more general class of Hamiltonians which contains, in particular, both, the generalized Pauli-Fierz and Nelson Hamiltonians and their complex dilation transformations. (Recall that the Nelson Hamiltonian is defined in Appendix D.) We consider Hamiltonians of the form

$$H_g = H_{0g} + I_g, \tag{4.1}$$

where $g > 0$ is a coupling constant, $H_{0g} := H_{pg} + H_f$, with $H_{pg} := -\kappa\Delta + V_g(x), \kappa \in \mathbb{C}, \kappa \neq 0$, and $I_g := g\sum_{1\leq m+n\leq 2} W_{m,n}$. We assume that $V_g(x)$ is Δ-bounded with the relative bound less than $|\kappa|/2$, more precisely, that it obeys the bound

$$\|V_g\psi\| \leq \frac{|\kappa|}{2}\|\Delta\psi\| + \|\psi\|, \tag{4.2}$$

uniformly in $g \leq 1$, where we set the constant in front of the second term on the r.h.s. to 1. This constant plays no role in our analysis. Moreover, we assume that the operators $W_{m,n}$ are of the form

$$W_{m,n} := \int_{\mathbb{R}^{3(m+n)}} \prod_1^{m+n}\left(\frac{dk_j}{|k_j|^{1/2}}\right)\prod_1^m a^*(k_j)w_{m,n}[k_1,\dots,k_{m+n}]\prod_{m+1}^{m+n} a(k_j), \tag{4.3}$$

where $w_{m,n}[k], k := (k_1,\dots,k_{m+n})$, the coupling functions, are operator-functions from $\mathbb{R}^{3(m+n)}$ to bounded operators on the particle space $\mathcal{H}_p$ obeying

$$\sup_{g\leq 1}\|w_{m,n}\|_\mu^{(0)} < \infty, \tag{4.4}$$

for some $\mu \geq 0$ and $\delta_0 > 0$ (the latter parameter is not displayed, see the next equation; also note that $w_{m,n}$ might depend on the coupling constant g). Here the norm $\|w_{m,n}\|_\mu^{(0)}$ is defined by

$$\|w_{m,n}\|_\mu^{(0)} := \sup_{|\delta|\leq\delta_0}\ \sup_{k\in\mathbb{R}^{3(m+n)}}\left\|\frac{e^{-\delta\langle x\rangle}w_{m,n}[k]e^{\delta\langle x\rangle}\langle p\rangle^{-(2-m-n)}}{[\min(\langle x\rangle^{m+n}\prod_1^{m+n}(|k_j|^{1/2}),1)]^\mu}\right\|_{part}. \tag{4.5}$$

Here $\|\cdot\|_{part}$ is the operator norm on the particle Hilbert space $\mathcal{H}_p$. We observe that for g sufficiently small

$$D(H_g) = D(H_0) \subset D(I_g).$$

We denote by GH_μ the class of (generalized particle-field) Hamiltonians satisfying the restrictions (4.1)–(4.5). We also denote by GH_μ^{mn} the class of operators of the form (4.3)–(4.5).

Clearly, both, the generalized Pauli-Fierz and Nelson, Hamiltonians belong to GH_μ with $\mu = 1/2$ for the generalized Pauli-Fierz Hamiltonian and $\mu > 0$ for the Nelson Hamiltonian and with $\kappa = 1/2$. Indeed, for the Nelson model, (D.1)–(D.5), $V_g = V$ obeys (4.2) and $w_{m,n}$ are 0 for $m+n=2$ and multiplication operators by the bounded functions $\kappa(k)e^{-ikx}$ and $\kappa(k)e^{ikx}$ for $m+n=1$. For the QED case (the generalized Pauli-Fierz Hamiltonian, (2.4)) V_g is given by (3.6) and $I_g := p\cdot A_1(x) + \frac{1}{2}g : A_1(x)^2 : +G(x)$, where the operator $G(x)$

is defined in (2.5).[8] From these expressions we see that V_g satisfies (4.2) and $w_{m,n}$ obey the conditions formulated above.

Dilation deformed generalized Pauli-Fierz and Nelson Hamiltonian also fit this framework. Let $H^{SM}_{g\theta}$ be a complex deformation of the QED Hamiltonian H^{SM}_g, i.e. the dilation transformation of the generalized Pauli-Fierz Hamiltonian H^{PF}_g. Then the operator $H_g := e^{\theta} H^{SM}_{g\theta}$ satisfies the restrictions imposed above with $\mu = 1/2$ and $\kappa = e^{-\operatorname{Re}\theta}/2$. For the Nelson model we have and $\mu > 0$.

5 Elimination of Particle and High-Photon Energy Degrees of Freedom

In this section we consider the operator families $H_g - \lambda$, where the operator $H_g = H_{g0} + I_g \in GH_\mu$ (see Sect. 4), and map them into families of operators acting on the Fock space only (elimination of the particle degrees of freedom). We will study properties of the latter operators in Sects. 7 and 8 after we introduce appropriate Banach spaces in Sect. 6.

Fix $j \geq 0$ and consider an eigenvalue $\lambda_j \in \sigma_d(H_{pg})$ and define

$$\delta_j := \operatorname{dist}(\lambda_j, (\sigma(H_{pg})/\{\lambda_j\}) + [0, \infty)). \tag{5.1}$$

We assume $\inf_{g\geq 0} \delta_j > 0$[9] and we define the set

$$Q_j := \left\{ \lambda \in \mathbb{C} \,\middle|\, \operatorname{Re}(\lambda - \lambda_j) \leq \frac{1}{3}\delta_j \text{ and } |\operatorname{Im}(\lambda - \lambda_j)| \leq \frac{1}{3}\delta_j \right\}. \tag{5.2}$$

Let P_{pj} be the orthogonal projection onto the eigenspace of H_{pg} corresponding to λ_j and, as usual, $\overline{P}_{pj} = \mathbf{1} - P_{pj}$. We define $H^{\delta}_{pg} := e^{-\varphi} H_{pg} e^{\varphi}$ and $P^{\delta}_{pj} := e^{-\varphi} P_{pj} e^{\varphi}$ with $\varphi = \delta\langle x\rangle$. We use the following parameter to measure the size of the resolvent of H^{δ}_{pg}:

$$\kappa_j^{-1} := \max\Big(\sup_{0\leq\delta\leq\overline{\delta}} \sup_{\lambda\in Q_j} \|(H^{\delta}_{pg} - \lambda)^{-1} \overline{P}^{\delta}_{pj}\|, \delta_j^{-1} \Big), \tag{5.3}$$

for $\overline{\delta} > 0$ sufficiently small. Note that if the operator H_{pg} is normal, as in the case of the problem of the ground state, where H_{pg} is self-adjoint, then κ_j can be easily estimated for δ_0 sufficiently small. If the operator H_{pg} is not normal, then getting an explicit upper bound on its resolvent requires some work. This will be done in the proof of Theorem 3.1 given in Sect. 9.

Our goal now is to define the renormalization map on the class generalized particle-field Hamiltonians GH_μ. This map is a composition of three maps which we introduce now. First of these is the smooth Feshbach-Schur map (SFM),[10] or decimation, map, F_π, which is

[8] Here the symbol $:W:$ stands for the standard Wick ordering of an operator W on our Fock space, i.e. in the expression for W in terms of the creation and annihilation operators, the creation operators are moved to the left and the annihilation ones, to the right.

[9] If $\inf_{g\geq 0} \delta_j = 0$ as it happens in the case when $\lambda_j|_{g=0}$ are degenerate, then in (5.1) we have to group the eigenvalues into clusters with the pairwise distances of order $O(1)$.

[10] In [5, 11, 12] this map is called the Feshbach map. As was pointed out to us by F. Klopp and B. Simon, the invertibility procedure at the heart of this map was introduced by I. Schur in 1917; it appeared implicitly in an independent work of H. Feshbach on the theory of nuclear reactions, in 1958, see [29] for further extensions and historical remarks.

defined as follows. We introduce a pair of almost projections

$$\pi \equiv \pi_j \equiv \pi[H_f] := P_{pj} \otimes \chi_{H_f \le \rho} \tag{5.4}$$

and $\overline{\pi} \equiv \overline{\pi}[H_f]$ which form a partition of unity $\pi^2 + \overline{\pi}^2 = \mathbf{1}$. Note that π and $\overline{\pi}$ commute with H_{0g} introduced in Sect. 4. Next, for $H_g = H_{0g} + I_g \in GH_\mu$, we define

$$H_{\bar{\pi}} := H_{0g} + \bar{\pi} I_g \bar{\pi}. \tag{5.5}$$

Finally, on the operators $H_g - \lambda$ s.t. $H_g = H_{0g} + I_g \in GH_\mu$ and

$$H_{\bar{\pi}} - \lambda \text{ is (bounded) invertible on } \operatorname{Ran} \bar{\pi}, \tag{5.6}$$

we define *smooth Feshbach-Schur map*, F_π, as

$$F_\pi(H_g - \lambda) := H_{0g} - \lambda + \pi I_g \pi - \pi I_g \bar{\pi}(H_{\bar{\pi}} - \lambda)^{-1} \bar{\pi} I_g \pi. \tag{5.7}$$

Observe that the last two operators on the r.h.s. are bounded since, for any operator I_g as described in Sect. 4,

$$I_g \pi \text{ and } \pi I_g \text{ extend to bounded operators on } \mathcal{H}.$$

Properties of the smooth Feshbach-Schur maps, used in this paper, are described in Appendix A. For more details see [5, 29].

Next, we introduce the *scaling transformation* $S_\rho : \mathcal{B}[\mathcal{H}] \to \mathcal{B}[\mathcal{H}]$, which acts on the particle component of $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{H}_f$ by identity and on the field one, by

$$S_\rho(\mathbf{1}) := \mathbf{1}, \qquad S_\rho(a^\#(k)) := \rho^{-d/2} a^\#(\rho^{-1}k), \tag{5.8}$$

where $a^\#(k)$ is either $a(k)$ or $a^*(k)$ and $k \in \mathbb{R}^3$.

Now, on Hamiltonians acting on $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{H}_f$ which are in the domain of the decimation map F_π we define the renormalization map $\mathcal{R}_{\rho j}$ as

$$\mathcal{R}_{\rho j} = \rho^{-1} S_\rho \circ F_\pi, \tag{5.9}$$

where recall $\pi \equiv \pi_j$. The parameter ρ here is the same as the one in (5.4). It gives a photon energy scale and it is restricted below.

To simplify the notation we assume that the eigenvalue λ_j of the operator H_{pg} is simple (otherwise we would have to deal with matrix-valued operators on $\mathcal{H}_f$). We have

Theorem 5.1 *Let H_g be a Hamiltonian of the class GH_μ defined in Sect. 4 with $\mu \ge 0$. We assume that* $\inf_{g\ge 0} \delta_j > 0$. *Then for $g \ll \rho \le \kappa_j$, $\rho \ge \delta_j/2$ and $\lambda \in Q_j$*

$$H_g - \lambda \in D(\mathcal{R}_{\rho j}). \tag{5.10}$$

Furthermore, $\mathcal{R}_{\rho j}(H_g - \lambda) = P_{pj} \otimes H_{\lambda j} + (H_{0g} - \lambda)(\bar{P}_{pj} \otimes \mathbf{1})$ where the family of operators $H_{\lambda j}$, acting on $\mathcal{F}$, is s.t. $H_{\lambda j} - H_f$ is bounded and analytic in $\lambda \in Q_j$.

A proof of Theorem 5.1 is similar to that of related results of [11–13]. We begin with

Proposition 5.2 *Let $g \ll \rho \leq \kappa_j$, $\rho \geq \delta_j/2$ and $\lambda \in Q_j$. Then the operators $H_{\overline{\pi}} - \lambda$ are invertible on* $\operatorname{Ran}\overline{\pi}$ *and we have the estimate*

$$\|\overline{\pi}(H_{\overline{\pi}} - \lambda)^{-1}\overline{\pi}\| \leq 4\rho^{-1}. \tag{5.11}$$

Proof First we show that for $\lambda \in Q_j$ the operator $H_{0g} - \lambda$ is invertible on $\operatorname{Ran}\overline{\pi}$ and the following estimate holds for $n = 0, 1$

$$\|\big(|p|^2 + H_f + 1\big)^n R_0(\lambda)\| \leq C\rho^{-1} \tag{5.12}$$

where $R_0(\lambda) := (H_{0g} - \lambda)^{-1}\overline{\pi}$. If the operator H_{pg} is self-adjoint then the estimates above are straightforward. In the non-self-adjoint case we proceed as follows.

Write $\overline{\pi} = P_{pj} \otimes \chi_{H_f \geq \rho} + \overline{P}_{pj} \otimes \mathbf{1}$, where, as usual, $\overline{P}_{pj} = \mathbf{1} - P_{pj}$. On $\operatorname{Ran}(P_{pj} \otimes \chi_{H_f \geq \rho})$ we have that $H_{0g} = \lambda_j + H_f$ and therefore, for $\lambda \in Q_j$,

$$\operatorname{Re}(H_{0g} - \lambda) = \operatorname{Re}(\lambda_j - \lambda) + H_f \geq -\frac{1}{3}\delta_j + \rho \geq \frac{1}{3}\rho.$$

Hence the operator $H_{0g} - \lambda$ is invertible on $\operatorname{Ran}(P_{pj} \otimes \chi_{H_f \geq \rho})$ for $\lambda \in Q_j$ and

$$\|(H_{0g} - \lambda)^{-1}(P_{pj} \otimes \chi_{H_f \geq \rho})\| \leq 3\rho^{-1}. \tag{5.13}$$

Next, $\sigma(H_{0g}|_{Ran(\overline{P}_{pj} \otimes \mathbf{1})}) = \sigma(H_{pg}|_{Ran\overline{P}_{pj}}) + \sigma(H_f) = (\sigma(H_{pg})/\{\lambda_j\}) + \bar{\mathbb{R}}^+$. Now, by the definition of Q_j we have $\inf_{s \geq 0} \operatorname{dist}(\lambda_j - s, Q_j) \leq \delta_j/2$. This and the definition of δ_j give

$$\operatorname{dist}(\sigma(H_{0g}|_{Ran(\overline{P}_{pj} \otimes \mathbf{1})}), Q_j) \geq \delta_j/2. \tag{5.14}$$

Therefore, for $\lambda \in Q_j$, the operator $H_{0g} - \lambda$ is invertible on $\operatorname{Ran}(\overline{P}_{pj} \otimes \mathbf{1})$. Since the operator $(H_{0g} - \lambda)^{-1}(\overline{P}_{pj} \otimes \mathbf{1})$ is analytic in a neighborhood of $\overline{Q_j}$ we have that $\sup_{\lambda \in Q_j} \|(H_{0g} - \lambda)^{-1}(\overline{P}_{pj} \otimes \mathbf{1})\| < \infty$.

We claim that

$$\sup_{\lambda \in Q_j} \|(H_{0g} - \lambda)^{-1}(\overline{P}_{pj} \otimes \mathbf{1})\| \leq \kappa_j^{-1} \tag{5.15}$$

where κ_j is defined in (5.3). Indeed, since the operator H_f is self-adjoint with the known spectrum, $[0, \infty)$, and since $Q_j = Q_j - [0, \infty)$, we can write, using the spectral theory,

$$\text{l.h.s. of (5.15)} = \sup_{\lambda \in Q_j} \|(H_{pg} - \lambda)^{-1}\overline{P}_{pj}\|. \tag{5.16}$$

Now, our claim follows from the definition (5.3) of κ_j.

Since $\rho \leq \kappa_j$, the inequalities (5.13) and (5.15) imply

$$\|R_0(\lambda)\| \leq 4\rho^{-1} \tag{5.17}$$

which implies (5.12) with $n = 0$ and $C = 4$.

The estimate (5.17) and the relation $H_{0g}R_0(\lambda) = \operatorname{Ran}\overline{\pi} + \lambda R_0(\lambda)$ imply the inequality $\|H_{0g}R_0(\lambda)\| \leq 2 + 4|e_0^{(p)}|/\rho$. Finally, since by (4.2), $\||p|^2\psi\| \leq 2\|H_{0g}\psi\| + 2\|\psi\|$, we have (5.12) with $n = 1$.

The inequality (5.12) implies the estimates

$$\left\|\langle p\rangle^{2-n}(H_f + 1)^{n/2}(H_{0g} - \lambda)^{-1}\bar{\pi}\right\| \leq C\rho^{-1}, \tag{5.18}$$

for $n=1,2$.

Now, we claim that

$$\|I_g(H_{0g}-\lambda)^{-1}\bar{\pi}\| \le Cg\rho^{-1}. \tag{5.19}$$

Indeed, let $f(k)$ be an operator-valued function on $\mathcal{H}_p$. Then we have the following standard estimates

$$\|a(f)\psi\| \le \left(\int \frac{\|f(k)\|^2_{part}}{|k|}d^3k\right)^{\frac{1}{2}} \|H_f^{1/2}\psi\| \tag{5.20}$$

(cf. (6.10) with $m+n=1$) and

$$\|a^*(f)\psi\|^2 = \int \|f(k)\|^2_{part} d^3k \|\psi\|^2 + \|a(f)\psi\|^2. \tag{5.21}$$

Equation (5.19) follows from the estimates (5.18), (5.20) and (5.21), the pull-through formula

$$a(k)f[H_f] = f[H_f+|k|]a(k), \tag{5.22}$$

and from the conditions on the operator I_g imposed in Sect. 4. For instance, for the term $W_{0,1}$ we have

$$\begin{aligned}
\|W_{1,0}\psi\| &\le \int_{|k|\le 1} \|w_{1,0}(k)\langle p\rangle^{-1}a^*(k)\langle p\rangle\psi\| \frac{d^3k}{\sqrt{|k|}} \\
&\le \left(\int_{|k|\le 1} \frac{\|w_{1,0}(k)\langle p\rangle^{-1}\|^2_{part}}{|k|}d^3k\right)^{\frac{1}{2}} \|\langle p\rangle\psi\| \\
&\quad + \left(\int_{|k|\le 1} \frac{\|w_{1,0}(k)\langle p\rangle^{-1}\|^2_{part}}{|k|^2}d^3k\right)^{\frac{1}{2}} \|H_f^{1/2}\langle p\rangle\psi\| \\
&\le \|w_{1,0}\|_\mu^{(0)} \|\langle p\rangle(H_f+1)^{1/2}\psi\|
\end{aligned} \tag{5.23}$$

for any $\mu > -1/2$. This together with (5.12) implies $\|W_{1,0}(H_{0g}-\lambda)^{-1}\bar{\pi}\| \le C\|w_{1,0}\|_\mu^{(0)}\rho^{-1}$. Now, the term $W_{0,2}$ is estimated as follows:

$$\begin{aligned}
\|W_{0,2}\psi\| &\le \int_{|k_1|\le 1}\int_{|k_2|\le 1} \|w_{0,2}(k_1,k_2)a(k_1)a(k_2)\psi\| \frac{d^3k_1}{\sqrt{|k_1|}}\frac{d^3k_2}{\sqrt{|k_2|}} \\
&\le \int_{|k_1|\le 1}\left(\int_{|k_2|\le 1} \frac{\|w_{0,2}(k_1,k_2)\|^2_{part}}{|k_2|}d^3k_2\right)^{\frac{1}{2}} \|H_f^{1/2}a(k_1)\psi\| \frac{d^3k_1}{\sqrt{|k_1|}} \\
&\le \int_{|k_1|\le 1}\left(\int_{|k_2|\le 1} \frac{\|w_{0,2}(k_1,k_2)\|^2_{part}}{|k_2|}d^3k_2\right)^{\frac{1}{2}} \|(H_f+|k_1|)^{1/2}a(k_1)\psi\| \frac{d^3k_1}{\sqrt{|k_1|}} \\
&= \int_{|k_1|\le 1}\left(\int_{|k_2|\le 1} \frac{\|w_{0,2}(k_1,k_2)\|^2_{part}}{|k_2|}d^3k_2\right)^{\frac{1}{2}} \|a(k_1)H_f^{1/2}\psi\| \frac{d^3k_1}{\sqrt{|k_1|}} \\
&\le \left(\int_{|k_1|\le 1}\int_{|k_2|\le 1} \frac{\|w_{0,2}(k_1,k_2)\|^2_{part}}{|k_1||k_2|}d^3k_1d^3k_2\right)^{\frac{1}{2}} \|H_f\psi\| \\
&\le \|w_{0,2}\|_\mu^{(0)} \|H_f\psi\|
\end{aligned} \tag{5.24}$$

for any $\mu > -1$.

Equation (5.19) implies that the series

$$\sum_{n=0}^{\infty} (H_{0g} - \lambda)^{-1} \big(\overline{\pi} I_g R_0(\lambda)\big)^n \tag{5.25}$$

converges absolutely on the invariant subspace $\mathrm{Ran}\overline{\pi}$, and is equal to $(H_{\tau_0 \overline{\pi}} - \lambda)^{-1}$, provided $g \ll \rho$. Estimating this series using (5.19) gives the desired estimate (5.11). □

Proof of Theorem 5.1 The last proposition together with the fact that the operators πI_g and $I_g \pi$ are bounded yields (5.6). The second part of the theorem follows from the definition of the Feshbach-Schur map, (5.7), the proposition and the Neumann series argument. □

Note that $K := \mathcal{R}_{\rho j}(H_g - \lambda) \restriction_{\mathrm{Ran}(\bar{P}_{pj} \otimes \mathbf{1})} = (H_{0g} - \lambda) \restriction_{\mathrm{Ran}(\bar{P}_{pj} \otimes \mathbf{1})}$ and therefore $\sigma(K) = \sigma(H_{pg})/\{\lambda_j\} + [0, \infty) - \lambda$. Hence for any $\lambda \in Q_j$ we have

$$\min\{|\mu - \lambda| \mid \mu \in (\sigma(H_{pg})/\{\lambda_j\}) + [0, \infty)\} \geq \delta_j - |\lambda - \lambda_j| \geq \frac{1}{2}\delta_j. \tag{5.26}$$

Therefore $0 \notin \sigma(K)$. This, the relation $\sigma(\mathcal{R}_{\rho j}(H_g - \lambda)) = \sigma(H_{\lambda j}) \cup \sigma(K)$ and Theorem A.1 of Appendix A imply

Corollary 5.3 *Let $\lambda \in Q_j$. Then $\lambda \in \sigma(H_g)$ if and only if $0 \in \sigma(H_{\lambda j})$. Similar statement holds also for point and essential spectra.*

This corollary shows that to describe the spectrum of the operator H_g in the domain Q_j it suffices to describe the spectrum of the operators $H_{\lambda j}$ which act on the smaller space $\mathcal{F}$. In the next section we introduce a convenient Banach space which contains the operators $H_{\lambda j}$ for $\lambda \in Q_j$.

Furthermore to prove bounds on resolvent in terms of bounds on $H_{\lambda j}^{-1}$ one uses the relation

$$\mathcal{R}_{\rho j}(H_g - \lambda)^{-1} = H_{\lambda j}^{-1}(P_{pj} \otimes \mathbf{1}) + (H_{0g} - \lambda)^{-1}(\bar{P}_{pj} \otimes \mathbf{1}). \tag{5.27}$$

6 A Banach Space of Hamiltonians

We construct a Banach space of Hamiltonians on which the renormalization transformation will be defined. In order not to complicate notation unnecessarily we will think about the creation- and annihilation operators used below as scalar operators, neglecting the helicity of photons. We explain at the end of Appendix C how to reinterpret the corresponding expression for the photon creation- and annihilation operators.

Let B_1^r denote the Cartesian product of r unit balls in $\mathbb{R}^3$, $I := [0, 1]$ and $m, n \geq 0$. Given functions $w_{m,n} : I \times B_1^{m+n} \to \mathbb{C}$, $m + n > 0$, we consider monomials, $W_{m,n} \equiv W_{m,n}[w_{m,n}]$, in the creation and annihilation operators defined as follows:

$$W_{m,n}[w_{m,n}] := \int_{B_1^{m+n}} \frac{dk_{(m,n)}}{|k_{(m,n)}|^{1/2}} a^*(k_{(m)}) w_{m,n}\big[H_f; k_{(m,n)}\big] a(\tilde{k}_{(n)}). \tag{6.1}$$

Furthermore for $w_{0,0} : [0, \infty) \to \mathbb{C}$ we define using the operator calculus $W_{0,0} := w_{0,0}[H_f]$ ($m = n = 0$). Here we are using the notation

$$k_{(m)} := (k_1, \dots, k_m) \in \mathbb{R}^{3m}, \qquad a^{\#}(k_{(m)}) := \prod_{i=1}^{m} a^{\#}(k_i), \tag{6.2}$$

$$k_{(m,n)} := (k_{(m)}, \tilde{k}_{(n)}), \qquad dk_{(m,n)} := \prod_{i=1}^{m} d^3 k_i \prod_{i=1}^{n} d^3 \tilde{k}_i, \tag{6.3}$$

$$|k_{(m,n)}| := |k_{(m)}| \cdot |\tilde{k}_{(n)}|, \qquad |k_{(m)}| := |k_1| \cdots |k_m|, \tag{6.4}$$

where $a^{\#}(k)$ stand for $a(k)$ either or $a^*(k)$. The notation $W_{m,n}[w_{m,n}]$ stresses the dependence of $W_{m,n}$ on $w_{m,n}$. Note that $W_{0,0}[w_{0,0}] := w_{0,0}[H_f]$.

We assume that, for every m and n with $m + n > 0$ and for $s \geq 1$, the function $w_{m,n}[r, , k_{(m,n)}]$ is s times continuously differentiable in $r \in I$, for almost every $k_{(m,n)} \in B_1^{m+n}$, and weakly differentiable in $k_{(m,n)} \in B_1^{m+n}$, for almost every r in I. As a function of $k_{(m,n)}$, it is totally symmetric w.r.t. the variables $k_{(m)} = (k_1, \dots, k_m)$ and $\tilde{k}_{(n)} = (\tilde{k}_1, \dots, \tilde{k}_n)$ and obeys the norm bound

$$\|w_{m,n}\|_{\mu,s} := \sum_{n=0}^{s} \|\partial_r^n w_{m,n}\|_{\mu} < \infty, \tag{6.5}$$

where $\mu \geq 0$, $s \geq 0$ and

$$\|w_{m,n}\|_{\mu} := \max_{j} \sup_{r \in I, k_{(m,n)} \in B_1^{m+n}} \big| |k_j|^{-\mu} w_{m,n}[r; k_{(m,n)}] \big|. \tag{6.6}$$

Here and in what follows $k_j \in \mathbb{R}^3$ is the j-th 3-vector in $k_{(m,n)}$ over which we take the supremum. For $m + n = 0$ the variable r ranges over $[0, \infty)$ and we assume that the following norm is finite:

$$\|w_{0,0}\|_{\mu,s} := |w_{0,0}(0)| + \sum_{1 \leq n \leq s} \sup_{r \in [0,\infty)} |\partial_r^n w_{0,0}(r)|. \tag{6.7}$$

(This norm is independent of μ, but we keep this index for notational convenience.) The Banach space of functions $w_{m,n}$ of this type is denoted by $\mathcal{W}_{m,n}^{\mu,s}$.

We fix three numbers $\mu \geq 0$, $0 < \xi < 1$ and $s \geq 1$ and define the Banach space

$$\mathcal{W}^{\mu,s} \equiv \mathcal{W}_{\xi}^{\mu,s} := \bigoplus_{m+n \geq 0} \mathcal{W}_{m,n}^{\mu,s}, \tag{6.8}$$

with the norm

$$\|\underline{w}\|_{\mu,s,\xi} := \sum_{m+n \geq 0} \xi^{-(m+n)} \|w_{m,n}\|_{\mu,s} < \infty. \tag{6.9}$$

Clearly, $\mathcal{W}_{\xi'}^{\mu',s'} \subset \mathcal{W}_{\xi}^{\mu,s}$ if $\mu' \geq \mu$, $s' \geq s$ and $\xi' \leq \xi$.

Let $\chi_1(r) \equiv \chi_{r \leq 1}$ be a smooth cut-off function s.t. $\chi_1 = 1$ for $r \leq 9/10$, $= 0$ for $r \geq 1$ and $0 \leq \chi_1(r) \leq 1$ and $\sup |\partial_r^n \chi_1(r)| \leq 30$ $\forall r$ and for $n = 1, 2$. We denote $\chi_\rho(r) \equiv \chi_{r \leq \rho} := \chi_1(r/\rho) \equiv \chi_{r/\rho \leq 1}$ and $\chi_\rho \equiv \chi_{H_f \leq \rho}$.

The following basic bound, proven in [2], links the norm defined in (6.6) to the operator norm on $\mathcal{B}[\mathcal{F}]$.

Theorem 6.1 *Fix $m, n \in \mathbb{N}_0$ such that $m+n \geq 1$. Suppose that $w_{m,n} \in \mathcal{W}^{\mu,1}_{m,n}$, and let $W_{m,n} \equiv W_{m,n}[w_{m,n}]$ be as defined in (6.1). Then for all $\lambda > 0$*

$$\left\|(H_f+\lambda)^{-m/2} W_{m,n} (H_f+\lambda)^{-n/2}\right\| \leq \|w_{m,n}\|_0, \tag{6.10}$$

and therefore

$$\left\|\chi_\rho W_{m,n} \chi_\rho\right\| \leq \frac{\rho^{(m+n)(1+\mu)}}{\sqrt{m!n!}} \|w_{m,n}\|_0, \tag{6.11}$$

where $\|\cdot\|$ denotes the operator norm on $\mathcal{B}[\mathcal{F}]$.

Theorem 6.1 says that the finiteness of $\|w_{m,n}\|_0$ insures that $W_{m,n}$ defines a bounded operator on $\mathcal{B}[\mathcal{F}]$.

With a sequence $\underline{w} := (w_{m,n})_{m+n\geq 0}$ in $\mathcal{W}^{\mu,s}$ we associate an operator by setting

$$H(\underline{w}) := W_{0,0}[\underline{w}] + \sum_{m+n\geq 1} \chi_1 W_{m,n}[\underline{w}]\chi_1 \tag{6.12}$$

where we write $W_{m,n}[\underline{w}] := W_{m,n}[w_{m,n}]$. The r.h.s. of (6.12) are said to be in *generalized normal (or Wick-ordered) form* of the operator $H(\underline{w})$. Theorem 6.1 shows that the series in (6.12) converges in the operator norm and obeys the estimate

$$\left\|H(\underline{w}) - W_{0,0}(\underline{w})\right\| \underline{w}_1 \|_{\mu,0,\xi}, \tag{6.13}$$

for arbitrary $\underline{w} = (w_{m,n})_{m+n\geq 0} \in \mathcal{W}^{\mu,0}$ and any $\mu > -1/2$. Here $\underline{w}_1 = (w_{m,n})_{m+n\geq 1}$. Hence the linear map

$$H : \underline{w} \to H(\underline{w}) \tag{6.14}$$

takes $\mathcal{W}^{\mu,0}$ into the set of closed operators on Fock space $\mathcal{F}$. The following result is proven in [2].

Theorem 6.2 *For any $\mu \geq 0$ and $0 < \xi < 1$, the map $H : \underline{w} \to H(\underline{w})$, given in (6.12), is injective.*

Furthermore, we define the Banach space

$$\mathcal{W}^{\mu,s}_1 := \bigoplus_{m+n\geq 1} \mathcal{W}^{\mu,s}_{m,n}, \tag{6.15}$$

to be the set of all sequences $\underline{w}_1 := (w_{m,n})_{m+n\geq 1}$ obeying

$$\|\underline{w}_1\|_{\mu,s,\xi} := \sum_{m+n\geq 1} \xi^{-(m+n)} \|w_{m,n}\|_{\mu,s} < \infty. \tag{6.16}$$

We define the spaces $\mathcal{W}^{\mu,s}_{op} := H(\mathcal{W}^{\mu,s})$, $\mathcal{W}^{\mu,s}_{1,op} := H(\mathcal{W}^{\mu,s}_1)$ and $\mathcal{W}^{\mu,s}_{mn,op} := H(\mathcal{W}^{\mu,s}_{mn})$. Sometimes we display the parameter ξ as in $\mathcal{W}^{\mu,s}_{op,\xi} := H(\mathcal{W}^{\mu,s}_\xi)$. Theorem 6.2 implies that $\mathcal{W}^{\mu,s}_{op} := H(\mathcal{W}^{\mu,s})$ is a Banach space under the norm $\| H(\underline{w})\|_{\mu,s,\xi} := \|\underline{w}\|_{\mu,s,\xi}$. Similarly, the spaces $\mathcal{W}^{\mu,s}_{1,op}$ and $\mathcal{W}^{\mu,s}_{mn,op}$ are also Banach spaces in the corresponding norms.

In this paper we need and consider only the case $s = 1$. However, we keep the more general notation for convenience of references elsewhere.

7 The operator $\mathcal{R}_{\rho j}(H_g - \lambda)$

In this section we give a detailed description of the family of operators $H_{\lambda j} := \mathcal{R}_{\rho j}(H_g - \lambda)|_{\mathrm{Ran}(P_{pj}\otimes\mathbf{1})}$ (see Theorem 5.1). Here, recall, that P_{pj} denotes the projection on the particle eigenspace corresponding to the eigenvalue λ_j. We define the following polydisc in $\mathcal{W}_{op}^{\mu,s}$:

$$\mathcal{D}^{\mu,s}(\alpha,\beta,\gamma) := \Big\{ H(\underline{w}) \in \mathcal{W}_{op}^{\mu,s} \Big| |w_{0,0}(0)| \le \alpha, \sup_{r\in[0,\infty)} |\partial_r w_{0,0}(r) - 1| \le \beta,\ \|\underline{w}_1\|_{\mu,s,\xi} \le \gamma \Big\}, \tag{7.1}$$

for $\alpha,\beta,\gamma > 0$. Recall that $\underline{w}_1 := (w_{m,n})_{m+n\ge1}$. In what follows we fix the parameter ξ in (7.1) as $\xi = 1/4$.

Theorem 7.1 *Let H_g be a Hamiltonian of the class GH_μ defined in Sect. 4 with $\mu \ge 0$. We assume that $\delta_j > 0$. Then for $g \ll \rho \le min(\kappa_j, 1/2)$ and $\lambda \in Q_j$,*

$$H_{\lambda j} - \rho^{-1}(\lambda_j - \lambda) \in \mathcal{D}^{\mu,s}(\alpha,\beta,\gamma), \tag{7.2}$$

where $\alpha = O(g^2\rho^{\mu-2})$, $\beta = O(g^2\rho^{\mu-1})$, $\gamma = O(g\rho^\mu)$.

Note that if $\psi_j^{(p)}$ is an eigenfunction of H_{pg} with the eigenvalue λ_j and $\Psi_j := \psi_j^{(p)} \otimes \Omega$, then we have

$$\lambda_j - \lambda = \langle H_g - \lambda\rangle_{\Psi_j}.$$

The proof of Theorem 7.1 follows the lines of the proof of Theorem IV.3 of [26]. It is similar to the proofs of related results of [11–13]. However, there are a few differences here. The main ones are that we have to deal with unbounded interactions and, more importantly, with momentum-anisotropic spaces. Since the proof of Theorem 7.1 is technically rather involved, it is delegated to an Appendix B.

8 Spectrum of H_g

In this section we describe the spectrum of the operator $H_g \in GH_\mu$ defined in Section 4. We begin with some definitions. Recall that $D(\lambda, r) := \{z \in \mathbb{C} | |z-\lambda| \le r\}$, a disc in the complex plane. Denote $\mathcal{D} := \mathcal{D}^{\mu,1}(\alpha,\beta,\gamma)$ with $\alpha,\beta,\gamma \ll 1$ and let $\mathcal{D}_s := \mathcal{D}^{\mu,1}(0,\beta,\gamma)$ (the subindex s stands for 'stable', not to be confused with the smoothness index s which in this section is taken to be 1). For $H \in \mathcal{D}$ we denote $H_u := \langle H\rangle_\Omega$ and $H_s := H - \langle H\rangle_\Omega\mathbf{1}$ (the unstable- and stable-central-space components of H, respectively). Note that if $H \in \mathcal{D}$, then $H_s \in \mathcal{D}_s$.

Recall that a complex function f from an open set $\mathcal{D}$ in a complex Banach space $\mathcal{B}$ is said to be *analytic* iff $\forall H \in \mathcal{D}$ and $\forall \xi \in \mathcal{B}$, $f(H+\tau\xi)$ is analytic in the complex variable τ for $|\tau|$ sufficiently small (see [15]). (One can show that f is analytic iff it is Gâteaux-differentiable ([15, 37]). A stronger notion of analyticity, requiring in addition that f is locally bounded, is used in [37].) Furthermore, if f is analytic in $\mathcal{D}$ and g is an analytic vector-function from an open set Ω in $\mathbb{C}$ into $\mathcal{D}$, then the composite function $f \circ g$ is analytic on Ω. In what follows $\mathcal{B}$ is the space of H_f-bounded operators on $\mathcal{F}$.

Our analysis uses the following result from [26]:

Theorem 8.1 *For α, β and γ sufficiently small there is an analytic map $e : \mathcal{D}_s \to D(0, 4\alpha)$ s.t. $e(H) \in \mathbb{R}$ for $H = H^*$ and for any $H \in \mathcal{D}_s$, $\sigma(H) \subset e(H) + S$, where*

$$S := \left\{ w \in \mathbb{C} \,\middle|\, |\operatorname{Im} w| \leq \frac{1}{3} \operatorname{Re} w \right\}. \tag{8.1}$$

Moreover, the number $e(H)$ is an eigenvalue of the operator H.

Let H_g be in the class GH_μ defined in Sect. 4 with $\mu > 0$ and let H_{zj} be the operator obtained from H_g according to Theorem 5.1. By Corollary 5.3, for $z \in Q_j$, we have that $z \in \sigma(H_g)$ if and only if $0 \in \sigma(H_{zj})$ and similarly for point and essential spectra. By Theorem 7.1, $\forall z \in Q_j$, $H_{zj} \in \mathcal{D}^{\mu,1}(\alpha, \beta, \delta)$ with $\alpha = O(g^2\rho^{-1})$, $\beta = O(g^2)$ and $\gamma = O(g\rho^\mu)$, where ρ satisfies $g \ll \rho \leq \min(\kappa_j, 1/2)$. Since by our assumption $g \ll 1$, we can choose ρ so that

$$g^2\rho^{-1}, g\rho^\mu \ll 1. \tag{8.2}$$

In this case the condition of Theorem 8.1 is satisfied for $H_{zjs} \in \mathcal{D}_s$. Therefore it is in the domain of the map $e : \mathcal{D}_s \to \mathbb{C}$ described in Theorem 8.1 above and we can define

$$\varphi_j(z) := E_j(z) + e(H_{zjs}), \tag{8.3}$$

where $E_j(z) := H_{zju} = \langle \Omega, H_{zj}\Omega \rangle$ and $z \in Q_j$. Let Γ_ρ be the unitary dilatation on $\mathcal{F}$ defined by

$$\Gamma_\rho = U_f(-\ln \rho) \tag{8.4}$$

where $U_f(-\ln \rho)$ is defined in Sect. 3. Our goal is to prove the following

Theorem 8.2 *Let the Hamiltonian H_g be in the class GH_μ defined in Sect. 4 with $\mu > 0$ and let* $\inf_{g \geq 0} \delta_j > 0$. *Then for $g \ll \kappa_j$,*

(i) *The equation $\varphi_j(\epsilon) = 0$ has a unique solution $e_j \in Q_j$ and this solution obeys the estimate $|e_j - \lambda_j| \leq 15\alpha$;*
(ii) *e_j is an eigenvalue of H_g and*

$$\sigma(H_g) \cap Q_j \subset \left\{ z \in Q_j \,\middle|\, \frac{1}{2} \operatorname{Re}(z - e_j) \geq |\operatorname{Im}(z - e_j)| \right\}; \tag{8.5}$$

(iii) *If ψ_j is an eigenfunction of the operator $H_{e_j j}$ corresponding to the eigenvalue* 0, *then the vector*

$$\Psi_j := Q_\pi\big(H_g - e_j\big)\Gamma_\rho^* \psi_j \neq 0, \tag{8.6}$$

where π and $Q_\pi(H)$ are defined in (5.4) *and* (A.1), *respectively, is an eigenfunction of the operator H_g corresponding to the eigenvalue e_j.*

Proof In this proof we omit the subindex j. (i) Since $e : \mathcal{D}_s \to D(0, 4\alpha)$ is an analytic map, $z \to H_{zs}$ is an analytic vector-function and $z \to E(z)$ is an analytic function on Q^{int}, by Theorem 5.1, we conclude that the function φ is analytic on Q^{int}. Here Q^{int} is the interior of the set Q.

Furthermore, the definitions (8.3) and $\Delta_0 E(z) := E(z) - \rho^{-1}(\lambda - z)$ (remember that in this proof $\lambda = \lambda_j$) imply that $\varphi(\lambda) = \Delta_0 E(\lambda) + e(H_{\lambda s})$.

We have, by Theorem 7.1, that $|\Delta_0 E(\lambda)| \leq \alpha$. Hence $|\varphi(\lambda)| \leq 5\alpha$. Furthermore since Q is inside a square in $\mathbb{C}$ of side $\delta/3$, we have, by the Cauchy formula, that

$$|\partial_z^m \Delta_0 E(z)| \leq \alpha(3/\delta)^m \quad \text{for } m = 0, 1 \tag{8.7}$$

(remember that in this proof $\delta = \delta_j$). Similarly we have:

$$|\partial_z e(H(z)_s)| \leq 4\alpha(3/\delta)^{-1}. \tag{8.8}$$

The last two inequalities and the equation $\Delta_0 E(z) := E(z) - \rho^{-1}(\lambda - z)$ give

$$|\partial_z \varphi(z) + 1| \leq 15\alpha/\delta. \tag{8.9}$$

Hence by inverse function theorem, for α sufficiently small the equation $\varphi(z) = 0$ has a unique solution, e, in Q and this solution satisfies the bound $|e - \lambda| \leq 15\alpha$.

(ii) By Theorem 8.1, $\varphi(z)$ is an eigenvalue of the operator $H_z = E(z) + H_{zs}$. Hence 0 is an eigenvalue of the operator H_e. By Corollary 5.3, z is an eigenvalue of $H_g \leftrightarrow 0$ is an eigenvalue of H_z. Hence e is an eigenvalue of the operator H_g.

Next, by Corollary 5.3, we have for any $z \in Q$

$$z \in \sigma(H_g) \leftrightarrow 0 \in \sigma(H_z). \tag{8.10}$$

Due to Theorem 8.1 we have that $\sigma(H_z) = E(z) + \sigma(H_{zs}) \subset E(z) + e(H_{zs}) + S = \varphi(z) + S$, where the set S is defined in (8.1). This together with (8.10) gives $z \in \sigma(H_g) \cap Q \leftrightarrow \varphi(z) \in -S$ or

$$\sigma(H_g) \cap Q = \varphi^{-1}(-S). \tag{8.11}$$

Now the second part of the proof will follow if we show that $\varphi^{-1}(-S)$ is a subset of the r.h.s. of (8.5). Denote $\mu := z - e$ and let

$$|\operatorname{Im} \mu| > \frac{1}{2}|\operatorname{Re} \mu|. \tag{8.12}$$

Let $w := -\varphi(z)$. Using that $\varphi(e) = 0$ and the integral of derivative formula we find

$$\varphi(z) = (z - e)g(z) \tag{8.13}$$

with $g(z) := \int_0^1 \varphi(e + s(z - e))ds$. This gives

$$|\operatorname{Im} w| = |\operatorname{Re} g \operatorname{Im} \mu + \operatorname{Im} g \operatorname{Re} \mu|. \tag{8.14}$$

Now, the definitions (8.3) and $\Delta_0 E(z) := E(z) - \rho^{-1}(\lambda - z)$ (remember that in this proof $\lambda = \lambda_j$) imply that

$$\partial_z \varphi(z) = -1 + \partial_z \Delta_0 E(z) + \partial_z e(H_{zs}). \tag{8.15}$$

This, the fact that $\overline{z} := e + s(z - e) \in Q$ and (8.7) and (8.8) give

$$|\operatorname{Re} g(\overline{z})| \geq 1 - O(\alpha) \quad \text{and} \quad |\operatorname{Im} g(\overline{z})| \leq O(\alpha). \tag{8.16}$$

Relations (8.14) and (8.16) imply the estimate

$$|\operatorname{Im} w| \geq (1 - O(\alpha))|\operatorname{Im} \mu| - O(\alpha)|\operatorname{Re} \mu|$$

which together with (8.12) gives

$$|\operatorname{Im} w| \geq \frac{1}{4}(1 - O(\alpha))|\operatorname{Im}\mu| + \frac{3}{8}(1 - O(\alpha))|\operatorname{Re}\mu|. \tag{8.17}$$

Similarly, we obtain

$$\begin{aligned} |\operatorname{Re} w| &= |\operatorname{Re} g \operatorname{Re}\mu - \operatorname{Im} g \operatorname{Im}\mu| \\ &\leq (1 + O(\alpha))|\operatorname{Re}\mu| + O(\alpha)|\operatorname{Im}\mu|. \end{aligned} \tag{8.18}$$

The last two relations imply $|\operatorname{Im} w| > \frac{1}{3}|\operatorname{Re} w|$ and therefore $w \notin S$ or what is the same $z \notin \varphi^{-1}(-S)$.

Now let $\operatorname{Re}\mu < 0$. Then (8.15)–(8.16) imply that $\operatorname{Re} w = -\operatorname{Re} g \operatorname{Re}\mu + \operatorname{Im} g \operatorname{Im}\mu = (-1 + O(\alpha))|\operatorname{Re}\mu| + O(\alpha \operatorname{Im}\mu)$. Thus, $\operatorname{Re} w = (-1 + O(\alpha))|\operatorname{Re}\mu|$, provided $|\operatorname{Im}\mu| \leq |\operatorname{Re}\mu|$. Hence also in this case we have $z \notin \varphi^{-1}(-S)$. Thus we conclude that $\varphi^{-1}(-S)$ is a subset of the set on the r.h.s. of (8.5), as claimed.

(iii) Finally, the last part of the theorem follows from Theorem A.1(iii) of Appendix A. Theorem 8.2 is proven. □

9 Proof of Theorems 1.1 and 1.2

We begin with the proof of existence and properties of the ground state. Note that the generalized Pauli-Fierz Hamiltonian H_g^{PF}, given in (2.9), is of the class GH_μ, $\mu > 0$ (see Sect. 4) and is self-adjoint. The operator H_g^{PF}, $g \ll \kappa_0$, clearly satisfies the conditions of Theorem 8.2 with $j = 0$. Hence it has a ground state for $g \ll \kappa_0$ with all the properties stated in Theorem 8.2. Moreover, the particle Hamiltonian H_p^{PF} entering H_g^{PF} is self-adjoint which implies that the constant κ_0, defined in (5.3), is $\kappa_0 = \operatorname{dist}(\sigma(H_p^{PF}|_{Ran \overline{P}_{p0}}), Q_j) \geq \delta_0/2$. Here, recall, $\delta_0 := \operatorname{dist}(\lambda_0, \sigma(H_p^{PF})/\{\lambda_0\})$, where λ_0 is the smallest eigenvalues of the operator H_p^{PF}. This implies the existence and properties of the ground state for H_g^{PF}, $g \ll \delta_0$.

Now, $H_p^{PF} = H_p + O(g^2)$ (see (2.10)). Hence, since we assumed that the eigenvalues of H_p are non-degenerate, the eigenvalues λ_j of the operator H_p^{PF} labeled in order of their increase satisfy $\lambda_j = \epsilon_j^{(p)} + O(g^2)$, where, recall, $\epsilon_j^{(p)}$ are the eigenvalues of the operator H_p given in (1.2). Therefore $\delta_0 = \epsilon_{gap}^{(p)}(\epsilon_0^{(p)}) + O(g^2)$, where, recall,

$$\epsilon_{gap}^{(p)}(\nu) := \min\{|\epsilon_i^{(p)} - \epsilon_j^{(p)}| \mid i \neq j, \epsilon_i^{(p)}, \epsilon_j^{(p)} \leq \nu\}.$$

Consequently, $g \ll \epsilon_{gap}^{(p)}(\epsilon_0^{(p)})$ implies $g \ll \kappa_0$ and therefore, since H_g^{PF} is unitary equivalent to H_g^{SM}, this proves the part of the statement of Theorem 1.1 concerning the ground state.

Note that the energy of the found ground state solves the equation $\varphi_0(\epsilon) = 0$ (see (8.3) for the definition of $\varphi_j(\epsilon)$).

Now we prove Theorem 3.1 which implies the part of the statement of Theorem 1.1 concerning the excited states and Theorem 1.2. Let $H_g := e^\theta H_{g\theta}^{PF}$ where $H_{g\theta}^{PF}$ is the complex deformation of the Hamiltonian H_g^{PF}, defined in Sect. 3. The Hamiltonian H_g belongs to the class GH_μ with $\mu > 0$. We will assume $0 < \operatorname{Im}\theta \leq \min(\theta_0, \pi/4)$, where θ_0 is defined in Condition (DA) of Sect. 1, $\operatorname{Re}\theta = 0$ and $g \ll \min(\kappa_j, \epsilon_{gap}^{(p)}(\nu))$, where κ_j is defined in (5.3).

Let H_p^{PF} and $H_{p\theta}^{PF}$ be the particle Hamiltonians entering H_g^{PF} and $H_{g\theta}^{PF}$, respectively. We show that the condition $\inf_{g \geq 0} \delta_j > 0$ of Sect. 5 is satisfied in the present case, provided

$j \le j(\nu)$, with $\nu < \inf \sigma_{ess}(H_p)$, and $g \ll \epsilon_{gap}^{(p)}(\nu)$. Here, recall, $j(\nu) := \max\{j : \epsilon_j^{(p)} \le \nu\}$ and $\epsilon_{gap}^{(p)}(\nu)$ is defined above. To do this we note first that, since $H_p^{PF} = H_p + O(g^2)$, we have $\nu < \inf \sigma_{ess}(H_p^{PF})$ for g sufficiently small. Furthermore, since we have chosen $\mathrm{Re}\,\theta = 0$ and since the eigenvalues λ_j of the operator $H_{pg} := e^\theta H_{p\theta}^{PF}$ satisfy $\lambda_j = e^\theta \epsilon_j^{PF}$, where ϵ_j^{PF} are eigenvalues of the operator $H_{p\theta}^{PF}$, we have that

$$\delta_j = \mathrm{dist}(\epsilon_j^{PF}, (\sigma(H_{p\theta}^{PF})/\{\epsilon_j^{PF}\}) + e^{-\theta}\overline{\mathbb{R}^+}).$$

Recall the expression $H_{p\theta}^{PF} = -\frac{1}{2}e^{-2\theta}\Delta + V_{g\theta}$ and recall that $V_{g\theta} = V_\theta + O(g^2)$ and that V_θ is Δ-compact, by Condition (V), which implies the Δ-compactness of V in the one particle case, and Condition (DA), which implies the Δ-compactness of V_θ in the one particle case. Then by the Balslev-Combes-Simon theorem, we have that $H_{p\theta}^{PF}$ has no complex eigenvalues in the domain $\{\mathrm{Re}\, z \le \nu\}$ and therefore its eigenvalues ϵ_j^{PF}, $j \le j(\nu)$, coincide with the eigenvalues of the operator H_p^{PF} which are $\le \nu$. Hence we have that

$$\delta_j = \min(\mathrm{dist}(\epsilon_j^{PF}, \sigma(H_p^{PF})/\{\epsilon_j^{PF}\}), (\epsilon_{j-1}^{PF} - \epsilon_j^{PF})\tan(\mathrm{Im}\,\theta)).$$

Since the eigenvalues $\epsilon_j^{(p)}$ are simple, we have that $\epsilon_j^{PF} = \epsilon_j^{(p)} + O(g^2)$ and therefore $\inf_{g\ge 0}\delta_j > 0$.

Thus, for any $j \le j(\nu)$, the operator $H_g (:= e^\theta H_{g\theta}^{PF})$, $g \ll \min(\kappa_j, \epsilon_{gap}^{(p)}(\nu))$, satisfies the conditions of Theorem 8.2. This implies that $H_{g\theta}^{PF}$, $g \ll \min(\kappa_j, \epsilon_{gap}^{(p)}(\nu))$, has an eigenvalue $\epsilon_j \in e^{-\theta} Q_j$ and that the spectrum of $H_{g\theta}^{PF}$ near $\epsilon_j^{PF} = e^{-\theta}\lambda_j$ is of the form

$$\sigma(H_{g\theta}^{PF}) \cap e^{-\theta} Q_j \subset \left\{ z \in e^{-\theta} Q_j \,\middle|\, \frac{1}{2}\mathrm{Re}(e^\theta(z - \epsilon_j)) \ge |\mathrm{Im}(e^\theta(z - \epsilon_j))| \right\}. \tag{9.1}$$

Here Q_j is given in (5.2) and can be rewritten in the present special case as

$$Q_j := \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Re}(e^\theta(z - \epsilon_j^{PF})) \le \frac{1}{3}\delta_j \text{ and } |\mathrm{Im}(e^\theta(z - \epsilon_j^{PF}))| \le \frac{1}{3}\delta_j \right\}. \tag{9.2}$$

Moreover, $e^\theta \epsilon_j$ is the unique solution to the equation $\varphi_j(\epsilon) = 0$ and $\epsilon_j \to \epsilon_j^{PF}$ as $g \to 0$.

Let $\varphi_j(\epsilon, \theta) \equiv \varphi_j(\epsilon)$ be the function constructed in (8.3) for the operator $H_g := e^\theta H_{g\theta}^{PF}$. It is not hard to see that $\varphi_j(\epsilon, \theta)$ is analytic in θ. Since by Theorem 8.2 $e^\theta \epsilon_j$ is a unique solution to the equation $\varphi_j(\epsilon, \theta) = 0$ we conclude that ϵ_j is analytic in θ (in the degenerate eigenvalue case, a fractional power of θ). On the other hand, by (3.4), ϵ_j is independent of $\mathrm{Re}\,\theta$. Hence it is independent of θ.

The eigenvalue ϵ_0 is always real and therefore is the eigenvalue also of H_g^{PF}. This is the ground state energy of H_g^{PF}. For $j > 0$ the eigenvalue ϵ_j can be either complex or real, i.e. either a resonance or an eigenvalue of H_g^{PF}. (If the (FGR) condition is satisfied then $\mathrm{Im}\,\epsilon_j < 0$ for $j \ne 0$ and, in fact, $\mathrm{Im}\,\epsilon_j = -\gamma_j g^2 + O(g^4)$ for some $\gamma_j > 0$ independent of g, see [13].) (In the degenerate case, the total multiplicity of the resonances and eigenvalues arising from ϵ_j^{PF} is equal to the multiplicity of ϵ_j^{PF}.)

Thus, since the r.h.s. of (9.1) $\subset S_j \equiv S_{j,k}$ defined in (1.8), we have proven all the statements of Theorem 3.1, but under the stronger assumption $g \ll \min(\kappa_j, \epsilon_{gap}^{(p)}(\nu))$. Now we relax this assumption.

Define $\delta_j^\# := \mathrm{dist}(\epsilon_j^{PF}, \sigma(H_p^{PF})/\{\epsilon_j^{PF}\})$. The following proposition states that the restrictions $g \ll \delta_j^\#$ and $|\mathrm{Im}\,\theta| \ll \delta_j^\#$ imply the restriction $g \ll \kappa_j$. Recall that κ_j and δ_j are defined in (5.3) and (5.1), respectively.

Proposition 9.1 *Assume that* $|\operatorname{Im}\theta| \ll \delta_j^{\#}$. *Then there is a numerical constant* $c > 0$ *s.t.* $\kappa_j \geq c\delta_j^{\#}\tan(\operatorname{Im}\theta)$.

Proof Observe first that this proposition concerns entirely the particle Hamiltonian $H_{pg} := e^{\theta} H_{p\theta}^{PF}$. In its proof we omit the subindices p and g.

First we estimate δ_j in terms of $\delta_j^{\#}$. We assume $\operatorname{Re}\theta = 0$. By the definitions of δ_j and of $H := e^{\theta} H_{\theta}^{PF}$ we have $\delta_j = \operatorname{dist}(\epsilon_j^{PF}, \sigma(H_{\theta}^{PF})/\{\epsilon_j^{PF}\} + e^{-\theta}\overline{\mathbb{R}^+})$. Since $\sigma(H_{\theta}^{PF}) = \{\epsilon_i^{PF}\} \cup e^{-2\theta}\overline{\mathbb{R}^+}$, this gives

$$\delta_j = \min[\operatorname{dist}(\epsilon_j^{PF}, \sigma(H^{PF})/\{\epsilon_j^{PF}\}), \operatorname{dist}(\epsilon_j^{PF}, \epsilon_{j-1}^{PF} + e^{-\theta}\overline{\mathbb{R}^+})]$$

which can be rewritten as

$$\delta_j = \min(\delta_j^{\#}, (\epsilon_j^{PF} - \epsilon_{j-1}^{PF})\tan(\operatorname{Im}\theta)). \tag{9.3}$$

This, in particular, gives $\delta_j^{\#} \geq \delta_j \geq \delta_j^{\#}\tan(\operatorname{Im}\theta)$.

Now we estimate the norm on the r.h.s. of (5.3). We begin with the case of $\delta = 0$. In what follows $\lambda \in Q_j$ is fixed. First, we write $\overline{P}_j = P_{<j} + P_{>j}$, where $P_{<j} := \sum_{i<j} P_i$ and $P_{>j} := \mathbf{1} - \sum_{i\leq j} P_i$. Here, recall, P_i are the eigenprojections of $H := e^{\theta} H_{\theta}^{PF}$ corresponding to the eigenvalues λ_i. Since $(H-\lambda)^{-1}P_{<j} = \sum_{i<j}(\lambda_i - \lambda)^{-1}P_i$, we have $\|(H-\lambda)^{-1}P_{<j}\| \leq C(\min_{i<j}|\lambda_i - \lambda|)^{-1}$. To estimate the r.h.s. of the above inequality we write for $\lambda \in Q_j$

$$\begin{aligned}\min_{i<j}|\lambda_i - \lambda| &\geq \min_{i<j}|\operatorname{Im}(\lambda_i - \lambda)| \\ &\geq \min_{i<j}|\operatorname{Im}(\lambda_i - \lambda_j)| - |\operatorname{Im}(\lambda_j - \lambda)|.\end{aligned}$$

By the definitions of δ_j and Q_j (see (5.1) and (5.2)) and by (9.3), we have $|\operatorname{Im}(\lambda_j - \lambda)| \leq \frac{1}{3}\delta_j \leq \frac{1}{3}(\epsilon_j^{PF} - \epsilon_{j-1}^{PF})\tan(\operatorname{Im}\theta))$. On the other hand, $|\operatorname{Im}(\lambda_i - \lambda_j)| = (\epsilon_j^{PF} - \epsilon_i^{PF})\sin(\operatorname{Im}\theta)$. Hence

$$\min_{i<j}|\lambda_i - \lambda| \geq (\epsilon_j^{PF} - \epsilon_{j-1}^{PF})\left(\sin(\operatorname{Im}\theta) - \frac{1}{3}\tan(\operatorname{Im}\theta)\right).$$

For $0 < \operatorname{Im}\theta \leq \pi/3$, this gives $\min_{i<j}|\lambda_i - \lambda| \geq \frac{1}{3}\delta_j^{\#}\sin(\operatorname{Im}\theta)$ for any $\lambda \in Q_j$. This, together with the estimate derived above, yields

$$\|(H-\lambda)^{-1}P_{<j}\| \leq C(\delta_j^{\#}\sin(\operatorname{Im}\theta))^{-1}. \tag{9.4}$$

To estimate $(H-\lambda)^{-1}P_{>j}$ we write it as the contour integral

$$(H-\lambda)^{-1}P_{>j} = \frac{1}{2\pi i}e^{-\theta}\oint_{\Gamma}\left(H_{\theta}^{PF} - z\right)^{-1}(z - e^{-\theta}\lambda)^{-1}dz, \tag{9.5}$$

where the contour Γ is defined as $\Gamma := \mu + i\mathbb{R}$, where $\mu := \frac{1}{4}\epsilon_j^{PF} + \frac{3}{4}\epsilon_{j+1}^{PF}$.

Next, expanding $e^{2\theta}V_g(e^{\theta}x)$ in θ, we have $H_{\theta}^{PF} = e^{-2\theta}H^{PF} + O(\theta)$. Hence for $|\operatorname{Im}\theta| \ll \inf_{z\in\Gamma}\operatorname{dist}(z, \sigma(H_{\theta}^{PF}))$ and $\operatorname{Re}\theta = 0$, this gives

$$\|(H_{\theta}^{PF} - z)^{-1}\| \leq 2\|(e^{-2\theta}H^{PF} - z)^{-1}\| \leq 2/\operatorname{dist}(z, \sigma(e^{-2\theta}H^{PF})).$$

Again, by $H_\theta^{PF} = e^{-2\theta} H^{PF} + O(\theta)$ and the condition $|\theta| \ll \inf_{z\in\Gamma} \mathrm{dist}(z, \sigma(H_\theta^{PF}))$, the spectrum of $e^{-2\theta} H^{PF}$ is at the distance $\ll \inf_{z\in\Gamma} \mathrm{dist}(z, \sigma(H_\theta^{PF}))$ from the spectrum of H_θ^{PF}. Using these estimates and using (9.5), we obtain

$$\|(H-\lambda)^{-1} P_{>j}\| \le \frac{1}{\pi} \oint_\Gamma [\mathrm{dist}(z, \sigma(H_\theta^{PF}))]^{-1} |z - e^{-\theta}\lambda|^{-1} dz. \tag{9.6}$$

We estimate the integrand on the r.h.s. of the above inequality. We have for $\lambda \in Q_j$

$$|e^\theta z - \lambda| \ge \sup_{s\ge 0}(|e^\theta z + s - \lambda_j| - |\lambda_j - s - \lambda|).$$

For $z \in \Gamma$, we have $\inf_{s\ge 0} |e^\theta z + s - \lambda_j| = |z - \epsilon_j^{PF}| = [(\frac{3}{4}(\epsilon_{j+1}^{PF} - \epsilon_j^{PF}))^2 + (\mathrm{Im}\, z)^2]^{1/2}$. Moreover, the definition of Q_j and (9.3) imply that $\sup_{\lambda\in Q_j} \inf_{s\ge 0} |\lambda_j - s - \lambda| \le \frac{1}{2}\delta_j \le \frac{1}{2}\delta_j^\#$. Combining the last three estimates we obtain

$$\inf_{\lambda\in Q_j} |e^\theta z - \lambda| \ge \frac{1}{8}(\delta_j^\# + |\mathrm{Im}\, z|). \tag{9.7}$$

Next, we have for $z \in \Gamma$, $\mathrm{dist}(z, \sigma(H_\theta^{PF})) = [(\epsilon_{j+1}^{PF} - (\frac{1}{4}\epsilon_j^{PF} + \frac{3}{4}\epsilon_{j+1}^{PF}))^2 + (\mathrm{Im}\, z)^2]^{1/2} = [(\frac{1}{4}(\epsilon_{j+1}^{PF} - \epsilon_j^{PF}))^2 + (\mathrm{Im}\, z)^2]^{1/2}$, which gives

$$\mathrm{dist}(z, \sigma(H_\theta^{PF})) \ge \frac{1}{8}(\delta_j^\# + |\mathrm{Im}\, z|). \tag{9.8}$$

If $|\mathrm{Im}\,\theta| \ll \delta_j^\#$, then estimates (9.6)–(9.8) give

$$\|(H-\lambda)^{-1} P_{>j}\| \le C(\delta_j^\#)^{-1}. \tag{9.9}$$

This together with the estimate (9.4) yields

$$\|(H-\lambda)^{-1}\| \le C(\delta_j^\# \sin(\mathrm{Im}\,\theta))^{-1}.$$

This gives the desired estimate of the norm on the r.h.s. of (5.3) for $\delta = 0$.

Now we explain how to modify the above estimate in order to bound the norm on the r.h.s. of (5.3) for $\delta > 0$. First we recall the definitions $H^\delta := e^{-\varphi} H e^{\varphi}$ and $P_j^\delta := e^{-\varphi} P_j e^{\varphi}$ with $\varphi = \delta\langle x\rangle$. By a standard result, for δ sufficiently small,

$$\sigma(H^\delta) \cap \{\mathrm{Re}\, z \le \nu\} = \sigma(H) \cap \{\mathrm{Re}\, z \le \nu\}.$$

This and the boundedness of P_j^δ show that the estimate (9.4) remains valid if we replace the operators H and $P_{<j}$ by the operators H^δ and $P_{<j}^\delta$.

Now to prove the estimate (9.9) with the operator H replaced by the operator H^δ we use in addition to the estimates above the estimate $\|R^\delta(z)\| \le 2\|R(z)\|$ for $z \in \Gamma$ which is proven as follows. By an explicit computation, $H^\delta = H + W$, where

$$W := e^\theta(-\nabla\varphi \cdot \nabla - \nabla \cdot \nabla\varphi - |\nabla\varphi|^2).$$

Hence for small δ (recall that $\varphi(x) := \delta\langle x\rangle$) the operator H^δ is a relatively small perturbation of the operator H. In particular, for $z \in \Gamma$, $\|R(z)W\| \le C\delta \le 1/2$ and $R^\varphi(z) := [1 - R(z)W]^{-1} R(z)$, where $R(z) = (H_{pg} - z)^{-1}$ and $R^\delta(z) = (H_{pg}^\delta - z)^{-1}$. Using the last two relations we estimate $\|R^\delta(z)\| \le 2\|R(z)\|$ for $z \in \Gamma$. This, as was mentioned above, implies the estimate (9.9) with the operators H and $P_{>j}$ replaced by the operators H^δ and $P^\delta_{>j}$. This completes the proof of the proposition. □

Since $\epsilon_j^{PF} = \epsilon_j^{(p)} + O(g^2)$, we have that $\delta_j^\# \ge \epsilon_{gap}^{(p)}(\nu) - O(g^2)$ for $j \le j(\nu) := \max\{j : \epsilon_j^{(p)} \le \nu\}$. Therefore the restriction $g \ll \min(\kappa_j, \epsilon_{gap}^{(p)}(\nu))$, used above, is implied by the restriction

$$g \ll \epsilon_{gap}^{(p)}(\nu),$$

imposed in Theorem 3.1. As was mentioned in Sect. 3, Theorem 3.1 and the Combes argument presented in the paragraph containing (1.6) imply Theorems 1.1 and 1.2, provided we choose θ to be g-independent and satisfying $0 < \operatorname{Im}\theta \ll \epsilon_{gap}^{(p)}(\nu)$. This completes the proof of Theorem 1.1.

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Appendix A: The Smooth Feshbach-Schur Map

In this appendix, we describe properties of the *isospectral smooth Feshbach-Schur map* introduced in Sect. 5. In what follows $H_g = H_{0g} + I_g \in GH_\mu$ and we use the definitions of Sect. 5.

We define the following maps appearing in some identities below:

$$Q_\pi(H_g - \lambda) := \pi - \bar\pi(H_{\bar\pi} - \lambda)^{-1}\bar\pi I_g \pi, \tag{A.1}$$

$$Q_\pi^\#(H_g - \lambda) := \pi - \pi I_g \bar\pi(H_{\bar\pi} - \lambda)^{-1}\bar\pi. \tag{A.2}$$

Note that $Q_\pi(H_g - \lambda) \in \mathcal{B}(\operatorname{Ran}\pi, \mathcal{H})$ and $Q_\pi^\#(H_g - \lambda) \in \mathcal{B}(\mathcal{H}, \operatorname{Ran}\pi)$.

The following theorem, proven in [5] (see [29] for some extensions), states that the smooth Feshbach-Schur map of $H_g - \lambda$ is isospectral to $H_g - \lambda$.

Theorem A.1 *Let $H_g = H_{0g} + I_g$ satisfy* (5.6). *Then, as was mentioned in Sect.* 5, *the smooth Feshbach-Schur map F_π is defined on $H_g - \lambda$ and has the following properties*:

(i) $\lambda \in \rho(H_g) \Leftrightarrow 0 \in \rho(F_\pi(H_g - \lambda))$, *i.e. $H_g - \lambda$ is bounded invertible on $\mathcal{H}$ if and only if $F_\pi(H_g - \lambda)$ is bounded invertible on* $\operatorname{Ran}\chi$;

(ii) *If $\psi \in \mathcal{H} \setminus \{0\}$ solves $H_g\psi = \lambda\psi$ then $\varphi := \chi\psi \in \operatorname{Ran}\pi \setminus \{0\}$ solves $F_\chi(H_g - \lambda)\varphi = 0$*;

(iii) *If $\varphi \in \operatorname{Ran}\chi \setminus \{0\}$ solves $F_\pi(H_g - \lambda)\varphi = 0$ then $\psi := Q_\pi(H_g - \lambda)\varphi \in \mathcal{H} \setminus \{0\}$ solves $H_g\psi = \lambda\psi$*;

(iv) *The multiplicity of the spectral value* $\{0\}$ *is conserved in the sense that* $\dim\operatorname{Ker}(H_g - \lambda) = \dim\operatorname{Ker}F_\pi(H_g - \lambda)$;

(v) *If one of the inverses, $(H_g - \lambda)^{-1}$ or $F_{\tau,\pi}(H_g - \lambda)^{-1}$, exists then so does the other and these inverses are related by*

$$(H_g - \lambda)^{-1} = Q_\pi(H_g - \lambda)F_\pi(H_g - \lambda)^{-1}Q_\pi(H_g - \lambda)^\# + \bar\pi(H_{\bar\pi} - \lambda)^{-1}\bar\pi, \tag{A.3}$$

and

$$F_\pi(H_g - \lambda)^{-1} = \pi(H_g - \lambda)^{-1}\pi + \bar\pi(H_{0g} - \lambda)^{-1}\bar\pi.$$

Appendix B: Proof of Theorem 7.1

In this appendix we prove Theorem 7.1. As was mentioned in Sect. 7, the proof follows the lines of the proof of Theorem 4.3 of [26] (cf. Theorem 3.8 of [5] and Theorem 28 of [30]). It is similar to the proofs of related results of [11, 12]. We begin with some preliminary results.

Recall the notation $H_g = H_{0g} + I_g$ (see (4.1)). According to the definition ((5.7)) of the smooth Feshbach-Schur map, F_π, we have that

$$F_\pi\big(H_g-\lambda\big) = H_{0g} - \lambda + \pi I_g \pi - \pi I_g \bar{\pi}\big(H_{0g} - \lambda + \bar{\pi} I_g \bar{\pi}\big)^{-1} \bar{\pi} I_g \pi. \tag{B.1}$$

Here, recall, $\pi \equiv \pi[H_f]$ is defined in (5.4) and $\bar{\pi} \equiv \bar{\pi}[H_f] := \mathbf{1} - \pi[H_f]$. Note that, due to (5.19), the Neumann series expansion in $\bar{\pi} I_g \bar{\pi}$ of the resolvent in (B.1) is norm convergent and yields

$$F_\pi\big(H_g-\lambda\big) = H_{0g} - \lambda + \sum_{L=1}^{\infty}(-1)^{L-1}\pi I_g\big((H_{0g}-\lambda)^{-1}\bar{\pi}^2 I_g\big)^{L-1}\pi. \tag{B.2}$$

To write the Neumann series on the right side of (B.2) in the generalized normal form we use Wick's theorem, which we formulate now.

We begin with some notation. Recall the definition of the spaces GH_μ^{mn} in Sect. 4. For $W_{m,n} \in GH_\mu^{mn}$ of the form (4.3), we denote $W_{m,n} \equiv W_{m,n}[\underline{w}]$, where $\underline{w} := (w_{m,n})_{1 \le m+n \le 2}$ with $w_{m,n}$ satisfying (4.4) (not to confuse with the definitions of Sect. 6). We introduce the operator families

$$\begin{aligned} W_{p,q}^{m,n}\big[\underline{w}\big|k_{(m,n)}\big] :=& \int_{B_1^{p+q}} \frac{dx_{(p,q)}}{|x_{(p,q)}|^{1/2}} a^*(x_{(p)}) \\ &\times w_{m+p,n+q}\big[k_{(m)}, x_{(p)}, \tilde{k}_{(n)}, \tilde{x}_{(q)}\big] a(\tilde{x}_{(q)}), \end{aligned} \tag{B.3}$$

for $m+n \ge 0$ and a.e. $k_{(m,n)} \in B_1^{m+n}$. Here we use the notation for $x_{(p,q)}$, $x_{(p)}$, $\tilde{x}_{(q)}$, etc. similar to the one introduced in (3.2)–(3.4). For $m = 0$ and/or $n = 0$, the variables $k_{(0)}$ and/or $\tilde{k}_{(0)}$ are dropped out. Denote by S_m the group of permutations of m elements. Define the symmetrization operation as

$$w_{m,n}^{(\mathrm{sym})}[k_{(m,n)}] := \frac{1}{m!n!}\sum_{\pi \in S_m}\sum_{\tilde{\pi} \in S_n} w_{m,n}[k_{\pi(1)}, \dots, k_{\pi(m)}; \tilde{k}_{\tilde{\pi}(1)}, \dots, \tilde{k}_{\tilde{\pi}(n)}]. \tag{B.4}$$

Finally, below we will use the notation

$$\Sigma[k_{(m)}] := |k_1| + \cdots + |k_m|, \tag{B.5}$$

$$k_{(M,N)} = (k_{(m_1,n_1)}^{(1)}, \dots, k_{(m_L,n_L)}^{(L)}), \qquad k_{(m_\ell,n_\ell)}^{(\ell)} = (k_{(m_\ell)}^{(\ell)}, \tilde{k}_{(n_\ell)}^{(\ell)}), \tag{B.6}$$

$$r_\ell := \Sigma[\tilde{k}_{(n_1)}^{(1)}] + \cdots + \Sigma[\tilde{k}_{(n_{\ell-1})}^{(\ell-1)}] + \Sigma[k_{(m_{\ell+1})}^{(\ell+1)}] + \cdots + \Sigma[k_{(m_L)}^{(L)}], \tag{B.7}$$

$$\tilde{r}_\ell := \Sigma[\tilde{k}_{(n_1)}^{(1)}] + \cdots + \Sigma[\tilde{k}_{(n_\ell)}^{(\ell)}] + \Sigma[k_{(m_{\ell+1})}^{(\ell+1)}] + \cdots + \Sigma[k_{(m_L)}^{(L)}], \tag{B.8}$$

with $r_\ell = 0$ if $n_1 = \cdots n_{\ell-1} = m_{\ell+1} = \cdots m_L = 0$ and similarly for $\tilde{r}_\ell$ and $m_1 + \cdots + m_L = M, n_1 + \cdots + n_L = N$.

Theorem B.1 (Wick Ordering) *Let $W_{m,n} \in GH_\mu^{mn}, m+n \geq 1$ and $F_j \equiv F_j[H_f], j = 0, \ldots, L$, where $F_j[r]$ are operators on the particle space which are C^s functions of r and satisfy the estimates $\|\langle p\rangle^{-2+n} F_j[r]\langle p\rangle^{-n}\| \leq C$ for $n = 0, 1, 2$. Write $W := \sum_{m+n\geq 1} W_{m,n}$ with $W_{m,n} := W_{m,n}[w_{m,n}]$. Then*

$$F_0 W F_1 W \cdots W F_{L-1} W F_L = P_{pj} \otimes \widetilde{W}, \tag{B.9}$$

where $\widetilde{W} := \widetilde{W}[\underline{\tilde{w}}]$, $\underline{\tilde{w}} := (\widetilde{w}_{M,N}^{(\mathrm{sym})})_{M+N\geq 0}$ with $\widetilde{w}_{M,N}^{(\mathrm{sym})}$ given by the symmetrization w.r.t. $k_{(M)}$ and $\tilde{k}_{(N)}$, of the coupling functions

$$\begin{aligned}
&\widetilde{w}_{M,N}[r; k_{(M,N)}] \\
&\quad = \sum_{\substack{m_1+\cdots+m_L=M,\\ n_1+\cdots+n_L=N}} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ m_\ell+p_\ell+n_\ell+q_\ell\geq 1}} \prod_{\ell=1}^{L} \left\{ \binom{m_\ell+p_\ell}{p_\ell} \binom{n_\ell+q_\ell}{q_\ell} \right\} \\
&\qquad \times F_0[r+\tilde{r}_0] \big\langle \psi_j^{(p)} \otimes \Omega \big| \widetilde{W}_1\big[k^{(1)}_{(m_1,n_1)}\big] F_1[H_f + r + \tilde{r}_1] \widetilde{W}_2\big[k^{(2)}_{(m_2,n_2)}\big] \\
&\qquad \times \cdots F_{L-1}[H_f + r + \tilde{r}_{L-1}] \widetilde{W}_L\big[k^{(L)}_{(m_L,n_L)}\big] \psi_j^{(p)} \otimes \Omega \big\rangle F_L[r+\tilde{r}_L],
\end{aligned} \tag{B.10}$$

with

$$\widetilde{W}_\ell\big[k_{(m_\ell,n_\ell)}\big] := W^{m_\ell,n_\ell}_{p_\ell,q_\ell}[\underline{w}|k_{(m_\ell,n_\ell)}]. \tag{B.11}$$

The proof of this theorem mimics the proof of [12, Theorem A.4].

Next, we mention some properties of the scaling transformation. It is easy to check that $S_\rho(H_f) = \rho H_f$, and hence

$$S_\rho(\chi_\rho) = \chi_1 \quad \text{and} \quad \rho^{-1} S_\rho\big(H_f\big) = H_f, \tag{B.12}$$

which means that the operator H_f is a *fixed point* of $\rho^{-1}S_\rho$. Further note that $E \cdot \mathbf{1}$ *is expanded* under the scaling map, $\rho^{-1}S_\rho(E\cdot\mathbf{1}) = \rho^{-1}E\cdot\mathbf{1}$, at a rate ρ^{-1}. Furthermore,

$$\rho^{-1}S_\rho\big(W_{m,n}[\underline{w}]\big) = W_{m,n}\big[s_\rho(\underline{w})\big] \tag{B.13}$$

where the map s_ρ is defined by $s_\rho(\underline{w}) := (s_\rho(w_{m,n}))_{m+n\geq 0}$ and, for all $(m,n) \in \mathbb{N}_0^2$,

$$s_\rho(w_{m,n})\big[k_{(m,n)}\big] = \rho^{m+n-1} w_{m,n}\big[\rho k_{(m,n)}\big]. \tag{B.14}$$

As a direct consequence of Theorem B.1 and (5.7), (B.13)–(B.14) and (B.2), we have

Theorem B.2 *Let $\lambda \in Q_j$ so that $H_g - \lambda \in \mathrm{dom}(\mathcal{R}_\rho)$. Then $\mathcal{R}_\rho(H_g - \lambda)\restriction_{\mathrm{Ran}(P_{pj}\otimes\mathbf{1})} - \rho_0^{-1}(\lambda_j - \lambda) = H(\underline{\hat{w}})$ where the sequence $\underline{\hat{w}}$ is described as follows: $\underline{\hat{w}} = (\hat{w}_{M,N}^{(sym)})_{M+N\geq 0}$ with $\hat{w}_{M,N}^{(sym)}$, the symmetrization w.r.t. $k^{(M)}$ and $\tilde{k}^{(N)}$ (as in (B.4)) of the kernels*

$$\begin{aligned}
&\hat{w}_{M,N}[r; k_{(M,N)}] \\
&\quad = \rho^{M+N-1} \sum_{L=1}^{\infty} (-1)^{L-1}
\end{aligned}$$

$$\times \sum_{\substack{m_1+\cdots+m_L=M,\\ n_1+\cdots+n_L=N}} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ m_\ell+p_\ell+n_\ell+q_\ell\ge 1}} \prod_{\ell=1}^{L} \left\{ \binom{m_\ell+p_\ell}{p_\ell} \binom{n_\ell+q_\ell}{q_\ell} \right\}$$
$$\times V_{\underline{m,p,n,q}}[r;k_{(M,N)}], \tag{B.15}$$

for $M+N\ge 1$, *and*

$$\hat{w}_{0,0}[r] = r + \rho^{-1} \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell\ge 1}} \prod_{\ell=1}^{L} V_{\underline{0,p,0,q}}[r] \tag{B.16}$$

for $M=N=0$. *Here* $\underline{m,p,n,q} := (m_1,p_1,n_1,q_1,\ldots,m_L,p_L,n_L,q_L) \in \mathbb{N}_0^{4L}$, *and*

$$V_{\underline{m,p,n,q}}[r;k_{(M,N)}] := \langle \psi_j^{(p)} \otimes \Omega, g^L F_0[H_f+r]$$
$$\times \prod_{\ell=1}^{L} \left\{ \widetilde{W}_\ell\big[\rho k^{(\ell)}_{(m_\ell,n_\ell)}\big] F_\ell[H_f+r] \right\} \psi_j^{(p)} \otimes \Omega \rangle \tag{B.17}$$

with $M := m_1+\cdots+m_L$, $N := n_1+\cdots+n_L$, $F_\ell[r] := P_{pj} \otimes \chi_1[r+\tilde{r}_\ell]$, *for* $\ell = 0, L$, *and*

$$F_\ell[r] := \bar{\pi}[\rho(r+\tilde{r}_\ell)]^2 \big(H_{pg} + \rho(r+\tilde{r}_\ell) - \lambda\big)^{-1}, \tag{B.18}$$

for $\ell = 1,\ldots,L-1$. *Here the notation introduced in* (B.3)–(B.8) *and* (B.11) *is used.*

We remark that Theorem B.2 determines $\underline{\hat{w}}$ only as a sequence of integral kernels that define an operator in $\mathcal{B}[\mathcal{F}]$. Now we show that $\underline{\hat{w}} \in \mathcal{W}^{\mu,s}$, i.e. $\|\underline{\hat{w}}\|_{\mu,s,\xi} < \infty$. In what follows we use the notation introduced in (B.3)–(B.8) and (B.11). To estimate $\underline{\hat{w}}$, we start with the following preparatory lemma

Lemma B.3 *Let* $\lambda \in Q_j$. *For fixed* $L \in \mathbb{N}$ *and* $\underline{m,p,n,q} \in \mathbb{N}_0^{4L}$, *we have* $V_{\underline{m,p,n,q}} \in \mathcal{W}^{\mu,s}_{M,N}$ *and*

$$\|V_{\underline{m,p,n,q}}\|_{\mu,s} \le \rho^{\mu+1} L^s \left(\frac{Cg}{\rho}\right)^L \prod_{\ell=1}^{L} \|w_{m_\ell+p_\ell,n_\ell+q_\ell}\|_\mu^{(0)}. \tag{B.19}$$

Proof First we derive the estimate (B.19) for $\mu = 0$. Recall that the operators $\widetilde{W}_\ell$ might be unbounded. To begin with, we estimate

$$\big|V_{\underline{m,p,n,q}}[r;k_{(M,N)}]\big| \le g^L \big\|F_0[H_f+r]\big\| \prod_{\ell=1}^{L} A_\ell, \tag{B.20}$$

where $A_\ell := \|\widetilde{W}_\ell[\rho k^{(\ell)}_{(m_\ell,n_\ell)}] F_\ell[H_f+r]\|$. We use the resolvents and cut-off functions hidden in the operators $F_\ell[H_f+r]$ in order to bound the creation and annihilation operators whenever they are present in $\widetilde{W}_\ell$.

Recall that the operator $F_\ell[H_f+r]$ we estimate below depends on λ, see (B.18). Now, we claim that for $\lambda \in Q_j$

$$\big\|(|p|^2 + \rho H_f + 1) F_\ell[H_f+r]\big\| \le C\rho^{-1} \tag{B.21}$$

for $\ell=1,\dots,L-1$ and $\|(|p|^2+H_f+1)F_L[H_f+r]\|\le C$. The last estimate is obvious. To prove the first estimate we use the inequality (4.2), in order to convert the operator $|p|^2$ into the operator H_{pg}:

$$\begin{aligned}&\big\|(|p|^2+\rho H_f+1)F_\ell[H_f+r]\big\|\\&\quad\le 2\big\|(H_{pg}+\rho(H_f+r+\tilde r_\ell)+3)F_\ell[H_f+r]\big\|.\end{aligned}$$

Clearly, it suffices to consider λ changing in sufficiently large bounded set. The above estimate gives

$$\big\|(|p|^2+\rho H_f+1)F_\ell[H_f+r]\big\|\le C\big\|F_\ell[H_f+r]\big\|+C. \tag{B.22}$$

If the operator $F_\ell[H_f+r]$ inside the operator norm on the r.h.s. is normal, as in the case of the ground state analysis, then its norm can be estimated in terms of its spectrum. For non-normal operators we proceed as follows. Using that $\bar\pi[H_f]:=P_{pj}\otimes\chi_{H_f\ge\rho}+\bar P_{pj}\otimes\mathbf{1}$, we write

$$F_\ell[H_f+r]:=P_{pj}\otimes[\chi_{s\ge\rho}]^2\big(\lambda_j+s-\lambda\big)^{-1}+\bar P_{pj}\otimes\mathbf{1}\big(\bar H_{pg}+s-\lambda\big)^{-1}, \tag{B.23}$$

where $s:=\rho(H_f+r+\tilde r_\ell)$, recall, $\bar P_{pj}:=\mathbf{1}-P_{pj}$ and $\bar H_{pg}:=H_{pg}\bar P_p$. Now, since $\mathrm{Re}(\lambda_j-\lambda)\ge-\rho/2\ge-s/2$ for $\lambda\in Q_j$ and $s\ge\rho$, we have that $\lambda_j+s-\lambda\ge\rho/2$ for the first term on the r.h.s. For the second term on the r.h.s., we observe that by the spectral decomposition of the operator s in (B.23) we have

$$\sup_{\lambda\in Q_j}\|(\bar P_{pj}\otimes\mathbf{1})(\bar H_{pg}+s-\lambda)^{-1}\|\le\sup_{\lambda\in Q_j,\mu\ge0}\|\bar P_{pj}(\bar H_{pg}+\mu-\lambda)^{-1}\|_{part}. \tag{B.24}$$

Since $Q_j-[0,\infty)=Q_j$ and due to (5.3) we have

$$\sup_{\lambda\in Q_j}\|(\bar P_{pj}\otimes\mathbf{1})(\bar H_{pg}+s-\lambda)^{-1}\|\le\sup_{\lambda\in Q_j}\|\bar P_{pj}(\bar H_{pg}-\lambda)^{-1}\|_{part}\le\kappa_j^{-1}. \tag{B.25}$$

Since $\rho\le\kappa_j$, the last estimate, together with the estimate of the first term on the r.h.s. of (B.23) mentioned above, yields $\|F_\ell[H_f+r]\|\le C\rho^{-1}$ for $\ell=1,\dots,L-1$. This, due to (B.22), implies the estimate (B.21).

Next, since $\widetilde W_\ell[\rho k^{(\ell)}_{(m_\ell,n_\ell)}]$ contain products of $p_\ell+q_\ell\le m_\ell+p_\ell+n_\ell+q_\ell\le2$ creation and annihilation operators (see (B.3) and (B.11) and the paragraph after (4.1)), we have, by (4.4), (5.20)–(5.23) and similar estimates (cf. (6.10)), that

$$\Big\|\widetilde W_\ell\big[\rho k^{(\ell)}_{(m_\ell,n_\ell)}\big]\langle p\rangle^{-(2-s_\ell)}(H_f+1)^{-s_\ell/2}\Big\|\le C\|w_{m'_\ell,n'_\ell}\|_0^{(0)}, \tag{B.26}$$

where $m'_\ell:=m_\ell+p_\ell$ and $n'_\ell:=n_\ell+q_\ell$ and $s_\ell:=m'_\ell+n'_\ell$ (remember that $s_\ell\le2$). Consequently,

$$A_\ell\le C\rho^{-1+\delta_{\ell,L}}\|w_{m'_\ell,n'_\ell}\|_0^{(0)}. \tag{B.27}$$

Now, since $\|F_0[H_f+r]\|_{\mathrm{op}}\le1$ we find from (B.20) and (B.26) that

$$\big|V_{\underline{m,p,n,q}}[r;k_{(M,N)}]\big|\le\rho\left(\frac{Cg}{\rho}\right)^L\prod_{\ell=1}^L\big\|w_{m_\ell+p_\ell,n_\ell+q_\ell}\big\|_0^{(0)} \tag{B.28}$$

and similarly for the r-derivatives. This proves the isotropic, (B.19) with $\mu = 0$, bound on the functions $V_{\underline{m,p,n,q}}[r; k_{(M,N)}]$.

Now we prove the anisotropic, $\mu > 0$, bound on $V_{\underline{m,p,n,q}}[r; k_{(M,N)}]$. Let $\varphi(x) := \delta\langle x\rangle$ for δ sufficiently small. Define for $\ell = 1, \dots, L-1$

$$F_\ell^\delta[H_f + r] := e^{-\varphi} F_\ell[H_f + r] e^{\varphi}$$

and

$$\widetilde{W}_\ell^\delta\big[k^{(\ell)}_{(m_\ell,n_\ell)}\big] := e^{-\varphi} \widetilde{W}_\ell\big[k^{(\ell)}_{(m_\ell,n_\ell)}\big] e^{\varphi}.$$

Note that this transformation effects only the particle variables.

Exactly in the same way as we proved the bounds (B.21), with $\ell = 1, \dots, L-1$, one can show the following estimates

$$\big\|(|p|^2 + \rho H_f + 1) F_\ell^\delta[H_f + r]\big\| \le C\rho^{-1}, \tag{B.29}$$

provided $\lambda \in Q_j$ and $\delta \le \delta_0$.

Now, expression (B.17) can be rewritten for any j as

$$\begin{aligned} V_{\underline{m,p,n,q}}[r; k_{(M,N)}] := g^L F_0[H_f + r] e^{\varphi} \prod_{\ell=1}^{j-1} \Big\{ \widetilde{W}_\ell^\delta\big[\rho k^{(\ell)}_{(m_\ell,n_\ell)}\big] F_\ell^\delta[H_f + r] \Big\} \\ \times e^{-\varphi} \widetilde{W}_j\big[\rho k^{(j)}_{(m_j,n_j)}\big] F_j[H_f + r] \prod_{\ell=j+1}^{L} \Big\{ \widetilde{W}_\ell\big[\rho k^{(\ell)}_{(m_\ell,n_\ell)}\big] F_\ell[H_f + r] \Big\}. \end{aligned}$$

Since, by the definition, the operator $F_0[H_f + r]$ contains the projection, P_p, we conclude that the operator $F_0[H_f + r]e^{\varphi}$ is bounded. Hence we obtain for $j = 1, \dots, L$

$$\big|V_{\underline{m,p,n,q}}[r; k_{(M,N)}]\big| \le C g^L \tilde{A}_j^\delta \prod_{\ell=1}^{j-1} A_\ell^\delta \prod_{\ell=j+1}^{L} A_\ell, \tag{B.30}$$

where $A_\ell^\delta := \|\widetilde{W}_\ell^\delta[\rho k^{(\ell)}_{(m_\ell,n_\ell)}] F_\ell^\delta[H_f + r]\|$ and $\tilde{A}_j^\delta := \|e^{-\varphi}\widetilde{W}_\ell[\rho k^{(\ell)}_{(m_\ell,n_\ell)}] F_\ell[H_f + r]\|$. Furthermore, since $\widetilde{W}_\ell^\delta[\rho k^{(\ell)}_{(m_\ell,n_\ell)}]$ contain products of $p_\ell + q_\ell \le 2$ creation and annihilation operators (see (B.3) and (B.11)), we have, by (4.4), (5.20)–(5.23) and similar estimates (cf. (6.10)), that

$$\Big\|\widetilde{W}_\ell^\delta\big[\rho k^{(\ell)}_{(m_\ell,n_\ell)}\big] \langle p\rangle^{-(2-s_\ell)} (H_f + 1)^{-s_\ell/2}\Big\| \le C \|w_{m'_\ell,n'_\ell}\|_0^{(0)} \tag{B.31}$$

and

$$\Big\|e^{-\varphi}\widetilde{W}_\ell\big[\rho k^{(\ell)}_{(m_\ell,n_\ell)}\big] \langle p\rangle^{-(2-s_\ell)} (H_f + 1)^{-s_\ell/2}\Big\| \le C |\rho k^{(\ell)}_{(m_\ell,n_\ell)}|^\mu \|w_{m'_\ell,n'_\ell}\|_\mu^{(0)}, \tag{B.32}$$

where $m'_\ell := m_\ell + p_\ell$ and $n'_\ell := n_\ell + q_\ell$ and $s_\ell := m'_\ell + n'_\ell$. Consequently,

$$A_\ell^\delta \le C\rho^{-1} \|w_{m'_\ell,n'_\ell}\|_0^{(0)} \quad \text{and} \quad \tilde{A}_j^\delta \le C\rho^{\mu-1} |k^{(j)}_{(m_j,n_j)}|^\mu \|w_{m'_j,n'_j}\|_\mu^{(0)}. \tag{B.33}$$

Putting the equations (B.30), (B.33) and (B.27) together we arrive at

$$\begin{aligned}
&\big|V_{\underline{m,p,n,q}}[r;k_{(M,N)}]\big| \\
&\quad\le \rho^{\mu+1}\left(\frac{Cg}{\rho}\right)^{L}|k^{(j)}_{(m_j,n_j)}|^{\mu}\,\|w_{m_j+p_j,n_j+q_j}\|^{(0)}_{\mu}\prod_{\ell\neq j}^{1,L}\|w_{m_\ell+p_\ell,n_\ell+q_\ell}\|^{(0)}_{0}
\end{aligned}\tag{B.34}$$

and similarly for the r-derivatives. Since any i, k_i is contained, as a 3-dimensional component, in $k^{(j)}_{(m_j,n_j)}$ for some j, we find (B.19). □

Proof of Theorem 7.1 As was mentioned above we present here only the case $s=1$, which is needed in this paper. Recall that we assume $\rho\le 1/2$ and we choose $\xi=1/4$. First, we apply Lemma B.3 to (B.15) and use that $\binom{m+p}{p}\le 2^{m+p}$. This yields

$$\begin{aligned}
\|\hat{w}_{M,N}\|_{\mu,s} &\le \sum_{L=1}^{\infty}\rho^{\mu}L^{s}\left(\frac{Cg}{\rho}\right)^{L}(2\rho)^{M+N} \\
&\quad\times\sum_{\substack{m_1+\cdots+m_L=M,\\ n_1+\cdots+n_L=N}}\ \sum_{\substack{p_1,q_1,\dots,p_L,q_L:\\ m_\ell+p_\ell+n_\ell+q_\ell\ge 1}}\ \prod_{\ell=1}^{L}\left\{2^{p_\ell+q_\ell}\|w_{m_\ell+p_\ell,n_\ell+q_\ell}\|^{(0)}_{\mu}\right\}.
\end{aligned}\tag{B.35}$$

Using the definition (6.16) and the inequality $2\rho\le 1$, we derive the following bound for $\underline{\hat{w}}_1:=(\hat{w}_{M,N})_{M+N\ge 1}$,

$$\begin{aligned}
\|\underline{\hat{w}}_1\|_{\mu,s,\xi} &:= \sum_{M+N\ge 1}\xi^{-(M+N)}\|\hat{w}_{M,N}\|_{\mu,s} \\
&\le 2\rho^{\mu+1}\sum_{L=1}^{\infty}L^{s}\left(\frac{Cg}{\rho}\right)^{L}\sum_{M+N\ge 1}\ \sum_{\substack{m_1+\cdots+m_L=M,\\ n_1+\cdots+n_L=N}}\ \sum_{\substack{p_1,q_1,\dots,p_L,q_L:\\ m_\ell+p_\ell+n_\ell+q_\ell\ge 1}} \\
&\quad\times\prod_{\ell=1}^{L}\left\{(2\xi)^{p_\ell+q_\ell}\xi^{-(m_\ell+p_\ell+n_\ell+q_\ell)}\|w_{m_\ell+p_\ell,n_\ell+q_\ell}\|^{(0)}_{\mu}\right\} \\
&\le 2\rho^{\mu+1}\sum_{L=1}^{\infty}L^{s}\left(\frac{Cg}{\rho}\right)^{L} \\
&\quad\times\left\{\sum_{m+n\ge 1}\left(\sum_{p=0}^{m}(2\xi)^{p}\right)\left(\sum_{q=0}^{n}(2\xi)^{q}\right)\xi^{-(m+n)}\|w_{m,n}\|^{(0)}_{\mu}\right\}^{L}.
\end{aligned}$$

Let $\|\underline{w}_1\|^{(0)}_{\mu,\xi}:=\sum_{M+N\ge 1}\xi^{-(m+n)}\|w_{m,n}\|^{(0)}_{\mu}$, where, recall, $\underline{w}_1:=(w_{m,n})_{m+n\ge 1}$ (we introduce this norm in order to ease the comparison with the results of [5]). Using the assumption $\xi=1/4$ and the estimate $\sum_{p=0}^{m}(2\xi)^p\le\sum_{p=0}^{\infty}(2\xi)^p=\frac{1}{1-2\xi}$, we obtain

$$\|\underline{\hat{w}}_1\|_{\mu,s,\xi}\le 2\rho^{\mu+1}\textstyle\sum_{L=1}^{\infty}L^{s}B^{L},\tag{B.36}$$

where

$$B:=\frac{Cg}{\rho(1-2\xi)^2}\|\underline{w}_1\|^{(0)}_{\mu,\xi}.\tag{B.37}$$

Our assumption $g \ll \rho$ also insures that $B \leq \frac{1}{2}$. Thus the geometric series on the r.h.s. of (B.36) is convergent. We obtain for $s = 0, 1$

$$\sum_{L=1}^{\infty} L^s B^L \leq 8B. \tag{B.38}$$

Inserting (B.38) into (B.36), we see that the r.h.s. of (B.36) is bounded by $16\rho^{\mu+1}B$ which, remembering the definition of B and the choice $\xi = 1/4$, gives

$$\|\underline{\hat{w}}_1\|_{\mu,s,\xi} \leq 64Cg\rho^{\mu}\|\underline{w}_1\|^{(0)}_{\mu,\xi}. \tag{B.39}$$

Next, we estimate $\hat{w}_{0,0}$. We analyze the expression (B.16). Using estimate (B.19) with $\underline{m} = 0, \underline{n} = 0$ (and consequently, $M = 0, N = 0$), we find

$$\rho^{-1}\|V_{\underline{0,p,0,q}}\|_{\mu,s} \leq L^s\rho^{\mu}\left(\frac{Cg}{\rho}\right)^L \prod_{\ell=1}^{L}\|w_{p_\ell,q_\ell}\|^{(0)}_{\mu}. \tag{B.40}$$

In fact, examining the proof of Lemma B.3 more carefully we see that the following, slightly stronger estimate is true

$$\rho^{-1}\sup_{r\in I}\left|\partial_r^s V_{\underline{0,p,0,q}}[r]\right| \leq L^s\rho^{\mu+s}\left(\frac{Cg}{\rho}\right)^L \prod_{\ell=1}^{L}\|w_{p_\ell,q_\ell}\|^{(0)}_{\mu}. \tag{B.41}$$

Now, using (B.41), we obtain

$$\begin{aligned}
&\rho^{-1}\sum_{L=2}^{\infty}\sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell\geq 1}}\sup_{r\in I}\left|\partial_r^s V_{\underline{0,p,0,q}}[r]\right| \\
&\quad\leq \rho^{s+\mu}\sum_{L=2}^{\infty}L^s\left(\frac{Cg}{\rho}\right)^L\left\{\sum_{p+q\geq 1}\|w_{p,q}\|^{(0)}_{\mu}\right\}^L \\
&\quad\leq \rho^{s+\mu}\sum_{L=2}^{\infty}L^s D^L,
\end{aligned}$$

where $D := Cg\xi\rho^{-1}\|\partial_r^s\underline{w}_1\|_{\mu,0,\xi}$ with, recall, $\underline{w}_1 := (w_{m,n})_{m+n\geq 1}$. Now, similarly to (B.38), using that $\sum_{L=2}^{\infty}L^s D^L \leq 12D^2$, for D satisfying $D \leq 1/2$ (recall $g \ll \rho$), we find for $s = 0, 1$

$$\rho^{-1}\sum_{L=2}^{\infty}\sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell\geq 1}}\sup_{r\in I}\left|\partial_r^s V_{\underline{0,p,0,q}}[r]\right| \leq 12\rho^{s+\mu}\left(\frac{Cg\xi}{\rho}\|\underline{w}_1\|^{(0)}_{\mu,\xi}\right)^2. \tag{B.42}$$

Next, (B.16) and (B.42) yield

$$\left|\hat{w}_{0,0}[0]\right| \leq 12\rho^{\mu}\left(\frac{Cg\xi}{\rho}\|\underline{w}_1\|^{(0)}_{\mu,\xi}\right)^2. \tag{B.43}$$

We find furthermore that

$$\sup_{r\in[0,\infty)} \left|\partial_r \hat{w}_{0,0}[r] - 1\right| \leq 12\rho^{\mu+1}\left(\frac{Cg\xi}{\rho}\|\underline{w}_1\|^{(0)}_{\mu,\xi}\right)^2. \tag{B.44}$$

Now, recall that $\|\underline{w}_1\|^{(0)}_{\mu,\xi} \leq C$ and $\xi = 1/4$. Hence (B.43), (B.44) and (B.39) give (7.2) with $s = 1, \alpha = 12\rho^{\mu}(\frac{Cg}{\rho})^2$, $\beta = 12\rho^{\mu+1}(\frac{Cg}{\rho})^2$ and $\gamma = C\rho^{\mu}g$. This implies the statement of Theorem 7.1. □

Appendix C: Background on the Fock Space, etc.

Let $\mathfrak{h}$ be either $L^2(\mathbb{R}^3, \mathbb{C}, d^3k)$ or $L^2(\mathbb{R}^3, \mathbb{C}^2, d^3k)$. In the first case we consider $\mathfrak{h}$ as the Hilbert space of one-particle states of a scalar Boson or a phonon, and in the second case, of a photon. The variable $k \in \mathbb{R}^3$ is the wave vector or momentum of the particle. (Recall that throughout this paper, the velocity of light, c, and Planck's constant, $\hbar$, are set equal to 1.) The Bosonic Fock space, $\mathcal{F}$, over $\mathfrak{h}$ is defined by

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{S}_n \mathfrak{h}^{\otimes n}, \tag{C.1}$$

where $\mathcal{S}_n$ is the orthogonal projection onto the subspace of totally symmetric n-particle wave functions contained in the n-fold tensor product $\mathfrak{h}^{\otimes n}$ of $\mathfrak{h}$; and $\mathcal{S}_0\mathfrak{h}^{\otimes 0} := \mathbb{C}$. The vector $\Omega := 1 \bigoplus_{n=1}^{\infty} 0$ is called the *vacuum vector* in $\mathcal{F}$. Vectors $\Psi \in \mathcal{F}$ can be identified with sequences $(\psi_n)_{n=0}^{\infty}$ of n-particle wave functions, which are totally symmetric in their n arguments, and $\psi_0 \in \mathbb{C}$. In the first case these functions are of the form, $\psi_n(k_1, \ldots, k_n)$, while in the second case, of the form $\psi_n(k_1, \lambda_1, \ldots, k_n, \lambda_n)$, where $\lambda_j \in \{-1, 1\}$ are the polarization variables.

In what follows we present some key definitions in the first case only limiting ourselves to remarks at the end of this appendix on how these definitions have to be modified for the second case. The scalar product of two vectors Ψ and Φ is given by

$$\langle \Psi, \Phi\rangle := \sum_{n=0}^{\infty} \int \prod_{j=1}^{n} d^3k_j \overline{\psi_n(k_1, \ldots, k_n)} \varphi_n(k_1, \ldots, k_n). \tag{C.2}$$

Given a one particle dispersion relation $\omega(k)$, the energy of a configuration of n *non-interacting* field particles with wave vectors $k_1, \ldots, k_n$ is given by $\sum_{j=1}^{n} \omega(k_j)$. We define the *free-field Hamiltonian*, H_f, giving the field dynamics, by

$$(H_f\Psi)_n(k_1, \ldots, k_n) = \left(\sum_{j=1}^{n} \omega(k_j)\right) \psi_n(k_1, \ldots, k_n), \tag{C.3}$$

for $n \geq 1$ and $(H_f\Psi)_n = 0$ for $n = 0$. Here $\Psi = (\psi_n)_{n=0}^{\infty}$ (to be sure that the r.h.s. makes sense we can assume that $\psi_n = 0$, except for finitely many n, for which $\psi_n(k_1, \ldots, k_n)$ decrease rapidly at infinity). Clearly that the operator H_f has the single eigenvalue 0 with the eigenvector Ω and the rest of the spectrum absolutely continuous.

With each function $\varphi \in \mathfrak{h}$ one associates an *annihilation operator* $a(\varphi)$ defined as follows. For $\Psi = (\psi_n)_{n=0}^{\infty} \in \mathcal{F}$ with the property that $\psi_n = 0$, for all but finitely many n, the vector $a(\varphi)\Psi$ is defined by

$$(a(\varphi)\Psi)_n(k_1,\dots,k_n) := \sqrt{n+1}\int d^3k\overline{\varphi(k)}\psi_{n+1}(k,k_1,\dots,k_n). \tag{C.4}$$

These equations define a closable operator $a(\varphi)$ whose closure is also denoted by $a(\varphi)$. Equation (C.4) implies the relation

$$a(\varphi)\Omega = 0. \tag{C.5}$$

The creation operator $a^*(\varphi)$ is defined to be the adjoint of $a(\varphi)$ with respect to the scalar product defined in (C.2). Since $a(\varphi)$ is anti-linear, and $a^*(\varphi)$ is linear in φ, we write formally

$$a(\varphi) = \int d^3k\overline{\varphi(k)}a(k), \qquad a^*(\varphi) = \int d^3k\varphi(k)a^*(k), \tag{C.6}$$

where $a(k)$ and $a^*(k)$ are unbounded, operator-valued distributions. The latter are well-known to obey the *canonical commutation relations* (CCR):

$$\left[a^\#(k), a^\#(k')\right] = 0, \qquad \left[a(k), a^*(k')\right] = \delta^3(k-k'), \tag{C.7}$$

where $a^\# = a$ or a^*.

Now, using this one can rewrite the quantum Hamiltonian H_f in terms of the creation and annihilation operators, a and a^*, as

$$H_f = \int d^3k a^*(k)\omega(k)a(k), \tag{C.8}$$

acting on the Fock space $\mathcal{F}$.

More generally, for any operator, t, on the one-particle space $\mathfrak{h}$ we define the operator T on the Fock space $\mathcal{F}$ by the following formal expression $T := \int a^*(k)ta(k)dk$, where the operator t acts on the k-variable (T is the second quantization of t). The precise meaning of the latter expression can obtained by using a basis $\{\phi_j\}$ in the space $\mathfrak{h}$ to rewrite it as $T := \sum_j \int a^*(\phi_j)a(t^*\phi_j)dk$.

To modify the above definitions to the case of photons, one replaces the variable k by the pair (k,λ) and adds to the integrals in k also the sums over λ. In particular, the creation and annihilation operators have now two variables: $a^\#_\lambda(k) \equiv a^\#(k,\lambda)$; they satisfy the commutation relations

$$\left[a^\#_\lambda(k), a^\#_{\lambda'}(k')\right] = 0, \qquad \left[a_\lambda(k), a^*_{\lambda'}(k')\right] = \delta_{\lambda,\lambda'}\delta^3(k-k'). \tag{C.9}$$

One can also introduce the operator-valued transverse vector fields by

$$a^\#(k) := \sum_{\lambda\in\{-1,1\}} e_\lambda(k)a^\#_\lambda(k),$$

where $e_\lambda(k) \equiv e(k,\lambda)$ are polarization vectors, i.e. orthonormal vectors in $\mathbb{R}^3$ satisfying $k \cdot e_\lambda(k) = 0$. Then in order to reinterpret the expressions in this paper for the vector (photon)—case one either adds the variable λ as was mentioned above or replaces, in appropriate places, the usual product of scalar functions or scalar functions and scalar operators by the dot product of vector-functions or vector-functions and operator valued vector-functions.

Appendix D: Nelson Model

In this appendix we describe the Nelson model describing the interaction of electrons with quantized lattice vibrations. The Hamiltonian of this model is

$$H_g^N = H_0^N + I_g^N, \tag{D.1}$$

acting on the state space, $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{F}$, where now $\mathcal{F}$ is the Fock space for phonons, i. e. spinless, massless Bosons. Here g is a positive parameter—a coupling constant—which we assume to be small, and

$$H_0^N = H_p^N + H_f, \tag{D.2}$$

where $H_p^N = H_p$ and H_f are given in (1.2) and (1.4), respectively, but, in the last case, with the scalar creation and annihilation operators, a and a^*, and where the interaction operator is $I_g^N := gI$ with

$$I := \int \frac{\kappa(k)d^3k}{|k|^{1/2}}\left\{e^{-ikx}a^*(k) + e^{ikx}a(k)\right\} \tag{D.3}$$

(we can also treat terms quadratic in a and a^* but for the sake of exposition we leave such terms out). Here, $\kappa = \kappa(k)$ is a real function with the property that

$$|\kappa(k)| \le \text{const} \times \min\{1, |k|^\mu\}, \tag{D.4}$$

with $\mu > 0$, and

$$\int \frac{d^3k}{|k|}|\kappa(k)|^2 < \infty. \tag{D.5}$$

In the following, κ is fixed and g varies. It is easy to see that the operator I is symmetric and bounded relative to H_0, with the zero relative bound (see [58] for the corresponding definitions). Thus H_g^N is self-adjoint on the domain of H_0 for arbitrary g. Of course, for the Nelson model we can take an arbitrary dimension $d \ge 1$ rather than the dimension 3.

The complex deformation of the Nelson Hamiltonian is defined as (first for $\theta \in \mathbb{R}$)

$$H_{g\theta}^N := U_\theta H_g^{SM} U_\theta^{-1}. \tag{D.6}$$

Under Condition (DA), there is a Type-A ([50]) family $H_{g\theta}^N$ of operators analytic in the domain $|\operatorname{Im}\theta| < \theta_0$, which is equal to (D.6) for $\theta \in \mathbb{R}$ and s.t. $H_{g\theta}^{N*} = H_{g\bar{\theta}}^N$,

$$H_{g\theta}^N = U_{\operatorname{Re}\theta} H_{gi\operatorname{Im}\theta}^N U_{\operatorname{Re}\theta}^{-1}. \tag{D.7}$$

Furthermore, $H_{g\theta}^N$ can be written as

$$H_{g\theta}^N = H_{p\theta}^N \otimes \mathbf{1}_f + e^{-\theta}\mathbf{1}_p \otimes H_f + I_{g\theta}^N, \tag{D.8}$$

where $H_{p\theta}^N := U_{p\theta} H_p^N U_{p\theta}^{-1}$ and $I_{g\theta}^N := U_\theta I_g^N U_\theta^{-1}$.

In the Nelson model case one can weaken the restriction on the parameter ρ to $\rho \gg g^2$. One proceeds as follows. Assume for the moment that the parameter λ is real. Then the operator R_0 is non-negative and, due to (5.13) and (6.10), with $m+n\le 1$, and the fact that the operator I is a sum of creation and annihilation operators, we have

$$\left\| R_0^{1/2} I_g R_0^{1/2} \right\| \le C\rho^{-1/2} g, \tag{D.9}$$

where $R_0^{1/2} := (H_{0g}-\lambda)^{-1/2}\overline{\pi}$. Hence the following series

$$\sum_{n=0}^{\infty} R_0^{1/2}\big(g R_0^{1/2} I_g R_0^{1/2}\big)^n R_0^{1/2}$$

is well defined, converges absolutely and is equal to $\overline{\pi}(H_{\overline{\pi}}-\lambda)^{-1}\overline{\pi}$. Estimating this series gives the desired estimate (4.4) in the case of real λ. For complex λ we proceed in the same way replacing the factorization $R_0 = R_0^{1/2} R_0^{1/2}$, we used, by the factorization $R_0 = |R_0|^{1/2} U |R_0|^{1/2}$, where $|R_0|^{1/2} := |H_{0g}-\lambda|^{-1/2}\bar{\pi}$ and U is the unitary operator $U := (H_{0g}-\lambda)^{-1}|H_{0g}-\lambda|$.

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